ON THE DISTANCE FORMULAE IN THREE DIMENSIONAL TAXICAB SPACE

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Abstract

In this paper, we give distance formulas between a point and a line, a point and a plane, and between two lines in three dimensional taxicab space.

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1 INTRODUCTION

The plane taxicab geometry has been studied and improved by some mathematicians, (see [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11]). The taxicab space with dimension three can be introduced using the taxicab metric in the three dimensional analytical space by

$$d_T(P_1,P_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

instead of the well known Euclidean metric

$$d_E(P_1,P_2) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{1/2}$$

where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. That is, the three dimensional taxicab space $\mathbb{R}^3_T$ is constructed by simply replacing the Euclidean distance function d_E by the taxicab distance function d_T.

Notice that, the taxicab distance d_T between the points P_1 and P_2 is the length of the shortest path from P_1 to P_2 composed of three line segments parallel to the coordinate axes (or perpendicular to the coordinate planes). In this note, we study taxicab analogues of some topics of the Euclidean space which include the concept of distance. These topics are distances of a point to a plane, and to a line, and distance between two lines.

2 DISTANCE FORMULAE

In the three dimensional taxicab space, points, lines and planes are the same with in the Euclidean case. It can be easily shown that if

$$d_T : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$$

is defined as

$$d_T((x_1,y_1,z_1),(x_2,y_2,z_2)) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

then $\mathbb{R}^3_T := (\mathbb{R}^3, d_T)$ forms a metric space.

Suppose l is a line through P_1 and P_2. If l has direction vector (p,q,r), then it is easy to show that

$$(|p| + |q| + |r|).d_E(P_1,P_2) = ((p^2+q^2+r^2)^{1/2}).d_T(P_1,P_2).$$

Lemma 1: The taxicab distance is invariant under all translations in $\mathbb{R}^3_T$.That is

$$t : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \ni \quad t(x,y,z) = (x+a,\ y+b,\ z+c), \quad a,b,c \in \mathbb{R}$$

does not change the distance between two points in $\mathbb{R}^3_T$.

Proof is trivial.

Proposition 2: Taxicab distance of a point $P = (x_0,y_0,z_0)$ to the plane

$\mathcal{P} : Ax + By + Cz + D = 0$ is

$$d_T(P,\mathcal{P}) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\max\{|A|,\ |B|,\ |C|\}}.$$

Proof : In the taxicab space $\mathbb{R}^3_T$, the distance from a point P to a plane $\mathcal{P}$ is defined as

$$\begin{aligned} d_T(P,\mathcal{P}) &= \text{The smallest of all the } d_T(P,X), \text{ where } X \in \mathcal{P} \\ &= \min_{X\in\mathcal{P}} \quad d_T(P,X). \end{aligned}$$

Let l_x, l_y, l_z denote the lines passing through P and parallel to $x,y,z-$axis, respectively. Let $P_1 = l_x \cap \mathcal{P}$, $P_2 = l_y \cap \mathcal{P}$ and $P_3 = l_z \cap \mathcal{P}$. Clearly,

at least one of the points P_1, P_2, P_3 exists since $\mathcal{P}$ can not be parallel to all three of the coordinate axes. These points can be expressed as $P_1 = (x_1, y_0, z_0)$, $P_2 = (x_0, y_2, z_0)$ and $P_3 = (x_0, y_0, z_3)$ if they exist. Thus,

$$\begin{aligned} d_T(P,\mathcal{P}) &= \min\{d_T(P,P_1),\ d_T(P,P_2),\ d_T(P,P_3)\} \\ &= \min\{|x_0-x_1|,\ |y_0-y_2|,\ |z_0-z_3|\} \\ &= \min\begin{cases} |x_0-x_1| & \text{if } PP_1 \parallel x\text{-axis} \\ |y_0-y_2| & \text{if } PP_2 \parallel y\text{-axis} \\ |z_0-z_3| & \text{if } PP_3 \parallel z\text{-axis}, \end{cases} \end{aligned} \qquad (*)$$

where $\parallel$ is used to show parallelness. According to the position of $\mathcal{P}$ the following three main cases are possible:

Case I : $A \neq 0$, $B \neq 0$, $C \neq 0$. In this case if PP_1 is parallel to the $x-$axis then $Ax_1 + By_0 + Cz_0 = 0$ and

$$|x_0 - x_1| = |x_0 - (-By_0 - Cz_0 - D)/A| = \left|\frac{Ax_0 + By_0 + Cz_0 + D}{A}\right|.$$

Similarly if PP_2 is parallel to the $y-$axis then $Ax_0 + By_2 + Cz_0 + D = 0$ and

$$|y_0 - y_2| = |y_0 - (-Ax_0 - Cz_0 - D)/B| = \left|\frac{Ax_0 + By_0 + Cz_0 + D}{B}\right| \quad ;$$

and if PP_3 is parallel to the $z-$axis then $Ax_0 + By_0 + Cz_3 + D = 0$ and

$$|z_0 - z_3| = |z_0 - (-Ax_0 - By_0 - D)/C| = \left|\frac{Ax_0 + By_0 + Cz_0 + D}{C}\right| \quad .$$

Case II : Only one of A, B, C is zero. Say $A = 0$, $B \neq 0 \neq C$. In the case P is parallel to the $x-$axis and therefore P_1 doesn't exist, but

$By_2 + Cz_0 D = 0$ and $By_0 + Cz_3 + D = 0$ for the points P_2 and P_3. Thus

$$|y_0 - y_2| = \left| y_0 - (\frac{-Cz_0 - D}{B}) \right| = \left| \frac{By_0 + Cz_0 + D}{B} \right|$$

$$|z_0 - z_3| = \left| z_0 - (\frac{-By_0 - D}{C}) \right| = \left| \frac{By_0 + Cz_0 + D}{C} \right| \quad .$$

Similar results can easily be obtained for the subcases $B = 0$, $A \neq 0 \neq C$ or $C = 0$, $A \neq 0 \neq B$.

Case III : Exactly two of A, B, C are zero. Say $A \neq 0$, $B = 0 = C$. In this case P_2 and P_3 don't exist and $Ax_0 + D = 0$ for the point P_1 , and therefore

$$|x_0 - x_1| = \left| x_0 - (\frac{-D}{A}) \right| = \left| \frac{Ax_0 + D}{A} \right| .$$

Similarly,

$$|y_0 - y_2| = \left| \frac{By_0 + D}{B} \right| \qquad \text{if} \quad B \neq 0, \ A = 0 = C \quad ,$$

$$|z_0 - z_3| = \left| \frac{Cz_0 + D}{C} \right| \qquad \text{if} \quad C \neq 0, \ A = 0 = B \quad .$$

Notice that the results obtained in all of the above cases together with (*) gives the required formula □

Corollary : Let $d_E(P, \mathcal{P})$ and $d_T(P, \mathcal{P})$ denote the Euclidean and Taxicab distances from a point P and the plane $\mathcal{P} : Ax+By+Cz+D = 0$. Then

$$d_T(P, \mathcal{P}) = ((\max\{|A|, |B|, |C|\})^{-1}).((A^2 + B^2 + C^2)^{1/2}).d_E(P, \mathcal{P}).$$

Proposition 3 Taxicab distance of a point $P_0 = (x_0,\, y_0,\, z_0)$ to the line l given by

$$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

is

$$d_T(P_0, l) = \min \begin{cases} |p^{-1}|\ [|q(x_0-a) - p(y_0-b)| + |r(x_0-a) - p(z_0-c)|] & \text{if} \quad p \neq 0 \\ |q^{-1}|\ [|p(y_0-b) - q(x_0-a)| + |r(y_0-b) - q(z_0-c)|] & \text{if} \quad q \neq 0 \\ |r^{-1}|\ [|p(z_0-c) - r(x_0-a)| + |q(z_0-c) - r(y_0-b)|] & \text{if} \quad r \neq 0 \end{cases}$$

Proof : The following three cases are possible since $(p, q, r) \neq (0, 0, 0)$:

Case I : If $p \neq 0$ then the line l isn't parallel to the $Oyz-$coordinate plane and the plane $\mathcal{P}_x$ determined by $x = x_0$ in $\mathbb{R}^3_T$. Let $\mathcal{P}_x \cap l := P_1 = (x_0, y_1, z_1)$. In this case, notice that the given line l can be expressed as intersection of the planes

$$q(x-a) - p(y-b) = 0$$
$$r(x-a) - p(z-c) = 0$$

which are parallel to $z-$axis and $x-$axis, respectively. From these equations it is easy to obtain that

$$y_1 = [q(x_0-a) + pb]p^{-1} \qquad \text{and} \qquad z_1 = [r(x_0-a) + pc]p^{-1}$$

Therefore

$$\begin{aligned} d_T(P_0, P_1) &= |y_0 - y_1| + |z_0 - z_1| \\ &= |y_0 - [(q(x_0-a) + pb]p^{-1}| + |z_0 - [r(x_0-a) + pc]p^{-1}| \\ &= |[p(y_0-b) - q(x_0-a)]p^{-1}| + |[p(z_0-c) - r(x_0-a)]p^{-1}| \\ &= |p^{-1}|\ [|q(x_0-a) - p(y_0-b)| + |r(x_0-a) - p(z_0-c)|] \end{aligned}$$

Case II : If $q \neq 0$, in a similar way in case I, consider the plane $\mathcal{P}_y$ with the equation $y = y_0$, and define $\mathcal{P}_y \cap l := P_2 = (x_2, y_0, z_2)$. In this case, the line l can be expressed as intersection of the planes

$$p(y-b) - q(x-a) = 0 \qquad \text{and} \qquad r(y-b) - q(z-c) = 0.$$

Since P_2 is on each of these planes

$$x_2 = [p(y_0 - b) + qa]q^{-1} \qquad \text{and} \qquad z_2 = [r(y_0 - b) + qc]q^{-1}.$$

Thus

$$\begin{aligned} d_T(P_0, P_2) &= |x_0 - x_2| + |z_0 - z_2| \\ &= |q^{-1}| \, [|p(y_0 - b) - q(x_0 - a)| + |r(y_0 - b) - q(z_0 - c)|]. \end{aligned}$$

Case III : If $r \neq 0$, let $\mathcal{P}_z$ be the plane with the equation $z = z_0$ and define $P_3 := l \cap \mathcal{P}_z = (x_3, y_3, z_0)$. Following the way in the above cases one obtains

$$d_T(P_0, P_3) = \left|r^{-1}\right| [|p(z_0 - c) - r(x_0 - a)| + |q(z_0 - c) - r(y_0 - b)|].$$

Consequently,

$$\begin{aligned} d_T(P_0, l) &= \min d_T(P_0, X), \qquad X \in l \\ &= \min \{d_T(P_0, P_i)\}, \qquad i = 1, 2, 3 \end{aligned}$$

which completes the proof □

Corollary Let $P_0 = (x_0, y_0, z_0)$ and

$$l : \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

be given point-line pair in the taxicab space $\mathbb{R}^3_T$. If $K = |q(x_0 - a) - p(y_0 - b)|$, $L = |r(x_0 - a) - p(z_0 - c)\ |$ and $M = |r(y_0 - b) - q(z_0 - c)|$. Then

$$d_T(P_0, l) = \min \begin{cases} |p^{-1}|\,(K + L) & \text{if } p \neq 0 \\ |q^{-1}|\,(K + M) & \text{if } q \neq 0 \\ |r^{-1}|\,(L + M) & \text{if } r \neq 0\ . \end{cases}$$

Proposition 4: Taxicab distance between any two lines given by

$$l \quad \ldots \quad \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} = \lambda$$

$$l' \quad \ldots \quad \frac{x-a'}{p'} = \frac{y-b'}{q'} = \frac{z-c'}{r'} = \mu$$

in $\mathbb{R}^3_T$ can be expressed as follows:

If l is parallel to l' then $d_T(l, l') = d_T(A, l')$, where $A = (a, b, c)$.

If l is not parallel to l' then

$$d_T(l, l') = \min \begin{cases} \left|c' - c + [(a' - a)(qr' - rq') + (b' - b)(rp' - pr')](pq' - qp')^{-1}\right| & \text{if } pq' \neq qp' \\ \left|a' - a + [(b' - b)(rp' - pr') + (c' - c)(pq' - qp')](qr' - rq')^{-1}\right| & \text{if } qr' \neq rq' \\ \left|b' - b + [(c' - c)(pq' - qp') + (a' - a)(qr' - rq')](rp' - pr')^{-1}\right| & \text{if } rp' \neq pr' \end{cases}$$

Proof : $d_T(l, l')$ =min.$d_T(X, X')$, $X \in l$, $X' \in l'$. If l is parallel to l' then one can take $(p, q, r) = (p', q', r')$ and $P = (a'+\mu p, b'+\mu q, c'+\mu r)$ for any point P on l', without loss of generality. Then it can be easily computed using the formula given by proposition 3 that $d_T(P, l') = d_T(A, l')$.

If l and l' are not parallel then at least one of

$$pq' - qp', \quad qr' - rq', \quad rp' - pr'$$

is not zero. Let $pq' - qp' \neq 0$. In this subcase, l and l' lie in the following planes both of which are parallel to $z-$axis but not parallel to each other

$$\begin{array}{lll} \mathcal{P} & ... & qx - py - qa + pb = 0 \\ \mathcal{P}' & ... & q'x - p'y - q'a' - p'b' = 0 \end{array}$$

respectively. Let l' denote the line of intersection of $\mathcal{P}$ and $\mathcal{P}'$. That is, $l' = \mathcal{P} \cap \mathcal{P}'$. Clearly l' is also parallel to $z-$axis, and intersets both l and l'. Now, let $X = l \cap l''$ and $X' = l' \cap l'$. Then, clearly $X' = (x, y, z + d)$ if $X = (x, y, z)$. Thus, the line segment XX' will be the most convenient taxicab way from l to l' and $d_T(X, X') = |d|$. Thus,

$$\begin{aligned} x &= a + \lambda p = a' + \mu p' \\ y &= b + \lambda q = b' + \mu q' \\ z &= c + \lambda r = c' + \mu r' - d \end{aligned}$$

since X and X' are on the lines l and l', respectively.

It follows from this system of linear equations that

$$d = c' - c + [(a' - a)(qr' - rq') + (b' - b)(rp' - pr')](pq' - qp')^{-1}$$

and therefore

$$d_T(X, X') = \left| c' - c + [(a' - a)(qr' - rq') + (b' - b)(rp' - pr')](pq' - qp')^{-1} \right| .$$

Similarly, one can easily calculate the translation lenghts parallel to x or $y-$axis for the remaining two subcases where $qr' - rq' \neq 0$ or $rp' - pr' \neq 0$, respectively, as follows:

$$d' = a' - a + [(b' - b)(rp' - pr') + (c' - c)(pq' - qp')](qr' - rq')^{-1}$$

and

$$d' = b' - b + [(c' - c)(pq' - qp') + (a' - a)(qr' - rq')](rp' - pr')^{-1} \ .$$

Finally

$$d_T(l, l') = \min \{|d| \ , \ \ |d'| \ , \ \ |d'|\}$$

gives the required result □

Corollary : Let

$$l \qquad ... \qquad \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

and

$$l' \qquad ... \qquad \frac{x-a'}{p'} = \frac{y-b'}{q'} = \frac{z-c'}{r'}$$

be two skew lines in $\mathbb{R}^3_T$, and let

$$\alpha = pq' - qp', \quad \beta = qr' - rq', \quad \gamma = rp' - pr'$$

then

$$d_T(l, l') = \min \begin{cases} |c - c' + [(a - a')\beta + (b - b')\gamma]\alpha^{-1}| & \text{if} \quad \alpha \neq 0 \\ |a - a' + [(b - b')\gamma + (c - c')\alpha]\beta^{-1}| & \text{if} \quad \beta \neq 0 \\ |b - b' + [(c - c')\alpha + (a - a')\beta]\gamma^{-1}| & \text{if} \quad \gamma \neq 0 \end{cases} \ .$$

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