ON THE NORM IN HIGHER DIMENSIONAL TAXICAB SPACES

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Abstract

In this work, a norm is given for higher dimensional taxicab spaces, which generalize the norm in two dimensional case. Then its properties are studied.

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1 INTRODUCTION

The taxicab plane geometry has been studied by many authors. For some of them see the references [3], [4], [5], [6], [7], [8],[9], [10] , [11] and [12]. Let $\alpha = (a_1, a_2) \in R^2$. It is shown in [3] that

$$\|\alpha\| = \sqrt{< \alpha, \alpha > +2\,|a_1 a_2|}$$

defines a norm in the taxicab plane. In this work firstly, the concept of norm is generalized for higher dimensional taxicab spaces, and main properties of this norm is studied. A taxicab trigonometric cosine function, $\cos_T \theta$, has been defined and studied in [1]. But, it is shown here that another cosine function is induced by the taxicab norm. In the last part, connections among three type of cosine functions are determined.

2 TAXICAB DISTANCE and NORM

Let $A = (a_1, a_2, ..., a_n)$ and $B = (b_1, b_2, ..., b_n)$ be any two points in R^n. Then the nonnegative real number defined by

$$d_T(A, B) = \sum_{i=1}^{n} |a_i - b_i|$$

is called **taxicab distance** between A and B. As it is well known, the standart inner product of A and B is defined as

$$< A, B >= \sum_{i=1}^{n} a_i b_i$$

and the **norm** of A as

$$\|A\| = \sqrt{< A, A >} = \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}$$

in the Euclidean spaces R^n.

Now, we define a norm for the taxicab spaces, using the above norm and the inner product.

Definition : Let $A = (a_1, a_2, ..., a_n) \in R^n$. Then the nonnegative real number

$$\|A\|_T = \left(< A, A > +2 \sum_{i=1}^{n-1} |a_i a_j| \right)^{1/2} , \qquad i < j = 2, 3, ..., n$$

is called **taxicab norm** of A.

The following proposition simplifies the statement of the taxicab norm.

Proposition 1 : If $A = (a_1, a_2, ..., a_n) \in R^n$ then

$$\|A\|_T = \sum_{i=1}^{n} |a_i| = \left(\sum_{i,j=1}^{n} |a_i a_j| \right)^{1/2} .$$

Proof: From the definition of $\|A\|_T$ we have

$$\begin{aligned} \|A\|_T &= \left(\|A\|^2 + 2 \sum_{i=1}^{n-1} |a_i a_j| \right)^{1/2} & , i < j = 2, 3, ..., n \\ &= \left((a_1^2 + a_2^2 + ... + a_n^2) + 2 \sum_{i=1}^{n-1} |a_i a_j| \right)^{1/2} & , i < j = 2, 3, ..., n \\ &= \left(\left(|a_1|^2 + |a_2|^2 + ... + |a_n|^2 \right) + 2 \sum_{i=1}^{n-1} |a_i| \, |a_j| \right)^{1/2} & , i < j = 2, 3, ..., n \\ &= \left((|a_1| + |a_2| + ... + |a_n|)^2 \right)^{1/2} & \\ &= |a_1| + |a_2| + ... + |a_n| \; = \sum_{i=1}^{n} |a_i| & , \qquad i = 1, 2, ..., n \end{aligned}$$

Furthermore, from the fourth row of the above proof

$$\begin{aligned}\|A\|_T &= \left(|a_1|^2+|a_2|^2+...+|a_n|^2+2\sum_{i=1}^{n-1}|a_ia_j|\right)^{1/2}, \quad i<j=2,3,...,n\\ &= \left(|a_1|^2+|a_2|^2+...+|a_n|^2+\sum_{i,j=1}^{n}|a_ia_j|\right)^{1/2}, \quad i\neq j\\ &= \left(\sum_{i,j=1}^{n}|a_ia_j|\right)^{1/2}.\end{aligned}$$

The following proposition gives us a connection between taxicab norm and the taxicab distance.

Corollary : $d_T(A,O)=\|A\|_T$.

Proof: Using the definition of taxicab distance and the proposition 1, we have

$$d_T(A,O)=\sum_{i=1}^{n}|a_i-0|=\sum_{i=1}^{n}|a_i|=\|A\|_T \ .$$

The following proposition shows that the taxicab norm satisfies usual properties:

Proposition 2 : Let $A=(a_1,a_2,...,a_n)$, $B=(b_1,b_2,...,b_n)$, $C=(c_1,c_2,...,c_n)\in R^n$ and $r\in R$. Then

(i) $\|A\|_T\geq 0$ and $(\|A\|_T=0 \iff A=0)$

(ii) $\|rA\|_T=|r|\,\|A\|_T$

(iii) $\|A+B\|_T\leq\|A\|_T+\|B\|_T$

(iv) $\|A-B\|_T\leq\|A\|_T+\|B\|_T$

(v) $\|A-B\|_T\geq\|A\|_T-\|B\|_T$

(vi) $\|A-B\|_T\leq\|A-C\|_T+\|C-B\|_T$

Proof : It is simple to verify that the assertions (i) and (ii) are valid.

(iii) $\|A+B\|_T = \sum\limits_{i=1}^{n} |a_i+b_i|$
$\leq \sum\limits_{i=1}^{n} |a_i| + \sum\limits_{i=1}^{n} |b_i| = \|A\|_T + \|B\|_T$

(iv) $\|A-B\|_T = \sum\limits_{i=1}^{n} |a_i-b_i|$
$= \sum\limits_{i=1}^{n} |a_i+(-b_i)|$
$\leq \sum\limits_{i=1}^{n} |a_i| + \sum\limits_{i=1}^{n} |b_i| = \|A\|_T + \|B\|_T$

(v) $\|A-B\|_T = \sum\limits_{i=1}^{n} |a_i-b_i|$
$\geq \sum\limits_{i=1}^{n} |a_i| - \sum\limits_{i=1}^{n} |b_i| = \|A\|_T - \|B\|_T$

(vi) $\|A-B\|_T = \|A-B+C-C\|_T$
$= \|(A-C)+(C-B\|_T$
$\leq \|A-C\|_T + \|C-B\|_T$

Proposition 3 (*Schwarz Inequality*) **:** If $A=(a_1,a_2,...,a_n)$, $B=(b_1,b_2,...,b_n)\in R^n$. Then

$$|< A,B >| \leq \|A\|_T \, \|B\|_T \ .$$

Proof : Notice that we take

$$< A,B >_T = < A,B > \ .$$

i) If $A=0$ then $< A,B >=0$ and $|< A,B >|=0$. Furthermore $\|A\|_T=0$ and $\|A\|_T \,.\, \|B\|_T=0$.

ii) If $A\neq 0$, then

$$\begin{aligned} |< A,B >| &= \left|\sum_{i=1}^{n} a_i b_i\right| \\ &\leq \sum_{i=1}^{n} |a_i b_i| = |a_1b_1|+|a_2b_2|+...+|a_nb_n| \end{aligned} \tag{1}$$

and

$$\begin{aligned} \|A\|_T \,.\, \|B\|_T &= \left(\sum_{i=1}^{n} |a_i|\right) . \left(\sum_{i=1}^{n} |b_i|\right) \\ &= (|a_1| + |a_2| + ... + |a_n|) \,.\, (|b_1| + |b_2| + ... + |b_n|) \qquad (2) \\ &= (|a_1 b_1| + ... + |a_n b_n|) + \sum_{i,j=1}^{n} |a_i|\,|b_i|\,,\ i \neq j \end{aligned}$$

Thus, from (1) and (2) one get

$$|< A, B >| \leq \|A\|_T \,.\, \|B\|_T \ .$$

3 GEOMETRICAL INTERPRETATION

It is well known that

$$< A, B >= \|A\| \,.\, \|B\| \,.\, \cos\theta, \qquad 0 \leq \theta \leq \pi \qquad (3)$$

in the Euclidean spaces. Now, consider the Schwarz Inequality

$$|< A, B >| \leq \|A\|_T \,.\, \|B\|_T$$

in the taxicab spaces. If A and B are nonzero vectors one get

$$\frac{|< A, B >|}{\|A\|_T \,.\, \|B\|_T} \leq 1$$

from the Schwarz Inequality. The last inequality can be expressed as

$$-1 \leq \frac{< A, B >}{\|A\|_T \,.\, \|B\|_T} \leq 1$$

which allows to define **taxicab cosine, tcos**θ, as follows :

$$\mathrm{t}\cos\theta = \frac{< A, B >}{\|A\|_T \,.\, \|B\|_T}, \qquad 0 \leq \theta \leq \pi \ .$$

Thus,

$$< A, B >= \|A\|_T \, . \, \|B\|_T \, .\mathrm{t}\cos\theta \tag{4}$$

and consequently, the inner product can be interpreted in the taxicab space, as in the Euclidean spaces, by the related norm.

Clearly, $t\cos\theta \neq \cos\theta$ and $|\mathrm{t}\cos\theta| \leq |\cos\theta|$. Furthermore,

$$\mathrm{t}\cos\theta = \frac{\|A\| \, . \, \|B\|}{\|A\|_T \, . \, \|B\|_T} \cos\theta$$

and

$$\mathrm{t}\cos\theta = \frac{(\sum\limits_{i=1}^{n} a_i^2)^{1/2} . (\sum\limits_{i=1}^{n} b_i^2)^{1/2}}{\sum\limits_{i=1}^{n} |a_i| \, . \, \sum\limits_{i=1}^{n} |b_i|} \cos\theta \ . \tag{5}$$

4 **tcos**θ *and TAXICAB TRIGONOMETRY*

The investigation of trigonometry in taxicab plane geometry has been suggested as a broad research problem by Krause [6]. The Taxicab trigonometry developed, by defining a taxicab trigonometric function and also the taxicab trigonometric functions $\cos_T \theta$, $\sin_T \theta$, $tg_T\theta$ and $\cot_T \theta$ in [1]. The following discussion shows that $\mathrm{t}\cos\theta$ differs, also, from the

$\cos_T \theta$.

Figure.1. The Taxicab trigonometric circle.

The taxicab cosine function ($\cos_T \theta$) defined in [1], using the taxicab distance, is as follows :

$$\cos_T \theta = \frac{x}{\|OP\|_T} = \frac{x}{r} = \begin{cases} 1 - \frac{2\theta}{\pi} & \text{if} \quad 0 \leq \theta \leq \pi \\ -3 + \frac{2\theta}{\pi} & \text{if} \quad \pi \leq \theta \leq 2\pi \end{cases}$$

Remember that $\cos \theta = \frac{x'}{r}$. The graphs of $y = \cos \theta$ and $y = \cos_T \theta$ are given figure.2.

Figure.2. Graphs of $\cos\theta$ and $\cos_T\theta$.

Clearly $\cos\theta \neq \cos_T\theta$, $|\cos_T\theta| \leq |\cos\theta|$ and $\left(\cos\theta = \cos_T\theta \Leftrightarrow \theta \in \left\{\frac{k\pi}{2} \ : \ k \in Z\right\}\right)$.

Now consider the taxicab space with $n = 2$. If $A = (a_1, a_2)$ and $B = (b_1, b_2)$ then it is shown in [2] that

$$< A, B >= \|A\|_T \,.\, \|B\|_T \,.\, \cos_T\theta - R_\theta \ , \qquad R_\theta \in \{0, \pm 2\,|a_1 b_2|\,, 2\,|a_2 b_1|\}$$

using the equality

$$< A, B >= \|A\|_T \,.\, \|B\|_T \ \mathrm{t}\cos\theta$$

one get

$$\mathrm{t}\cos\theta = \cos_T\theta - \frac{R_\theta}{(\|A\|_T \,.\, \|B\|_T)}$$

Consequently, $\mathrm{t}\cos\theta \neq \cos_T\theta$ in general. Finaly, considering Eq.(5) for $n = 2$ one gets

$$\mathrm{t}\cos\theta = \frac{\sqrt{a_1^2 + a_2^2}.\sqrt{b_1^2 + b_2^2}}{(|a_1| + |a_2|)\,(|b_1| + |b_2|)} \cos\theta.$$

Without loosing generality, one can take $b_1 \neq 0$, $b_2 = 0$ and obtain

$$\mathrm{t}\cos\theta = \frac{\cos\theta}{\frac{|a_1|}{\sqrt{a_1^2+a_2^2}} + \frac{|a_2|}{\sqrt{a_1^2+a_2^2}}} = \begin{cases} 1/(1 + tg\theta) & if \quad 0 \leq \theta \leq \pi/2 \\ 1/(-1 + tg\theta) & if \quad \pi/2 \leq \theta \leq \pi \\ -1/(1 + tg\theta) & if \quad \pi \leq \theta \leq 3\pi/2 \\ -1/(-1 + tg\theta) & if \quad 3\pi/2 \leq \theta \leq 2\pi \end{cases}$$

The graphs of $y = t\cos\theta$, $y = \cos_T\theta$ and $y = \cos\theta$ are given in figure.3.

Figure.3. Graoh of $t\cos\theta$.

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