# TAXICAB GEOMETRY: ANOTHER LOOK AT CONIC SECTIONS

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The following article is written in response to [1] Moser and Kramer. They attempt to answer a question posed by [2] Reynolds.

The article by [2] Reynolds defines the taxicab distance between $(a_1,a_2)$ and $(b_1,b_2)$ by $d(A,B) = |a_1-b_1| + |a_2-b_2|$. She then deduces the nature of circles, ellipses, and hyperbolas using definitions analogous to those of Euclidean geometry.

The article by [1] Moser and Kramer defines a parabola as the locus of points equidistant from a focus $(x_0,y_0)$ and a line (the directrix) of the form $\{(x,y) \mid Ax + By + C = 0\}$. The approach taken is not entirely satisfying since they do not attempt to justify their definition of a line.

In Euclidean geometry, a line is defined as the locus of points in a plane equidistant from two distinct points. As [2] Reynolds points out, this locus does not necessarily take the form $\{(x,y) \mid Ax + By + C = 0\}$.

For the purposes of this paper, we define a line to be the locus of points equidistant from two distinct points. We also define the distance between a point $P$ and the line l as the minimum of $d(Q,P)$ where $Q$ is any point on $\ell$. We define a parabola as the locus of points equidistant from a focus and a line (the directrix).

In Figure 1, the line $\ell$ is equidistant from $A(-2,4)$ and $B(2,2)$. The diagram pictures the parabola with focus $(5,4)$ and directrix $\ell$. Two more parabolas are shown in Figures 2 and 3. Note that in Figure 3 our definition of a line agrees with [1] Moser and Kramer. Therefore, the reader should not be surprised that of the three parabolas shown, only the one in Figure 3 is a parabola according to [1] Moser and Kramer.

In [2] Reynolds defines an ellipse by $\{P \in R_2 \mid d(P,A) + d(P,B) = c\}$ where $A$ and $B$ are two fixed points (foci) and $c$ is a constant. For a description of such ellipses, the reader is referred to the article [2] quoted above.

Let us define an ellipse of the second kind with respect to a given line $\ell$ (the directrix), a given point $F$ (the focus), and a given eccentricity $e(0 < e < 1)$. Then such an ellipse is defined by $\{P \varepsilon R_2 \ |d(P,F)/d(P,\ell) = e\}$ where $d(P,\ell)$ denotes the shortest distance from $P$ to $\ell$.

In Figure 4, the line $\ell$ is equidistant from $(-1,6)$ and $(3,-4)$. The diagram pictures the ellipse corresponding to the directrix $\ell$, focus $(1,4)$ and eccentricity $1/2$. It is left to the reader to show that this ellipse is a convex hexagon with vertices at $(1,7)$, $(-1,5)$, $(\frac{-3}{2},4)$, $(-1,11/3)$, $(1,3)$ and $(2,4)$. It is a simple matter to show that this ellipse does not have the form given by [2] Reynolds.

In this paper we started with the natural definition of a line as the locus of points equidistant from two distinct points. We showed how this affects the results obtained by [1] Moser and Kramer. Finally, we showed that an ellipse defined using a line, focus, and eccentricity is not equivalent to an ellipse using two foci. We conclude that equivalent definitions under an Euclidean norm may yield contradictory definitions when generalized to a taxicab norm.

## REFERENCES

1. Moser, J., and Kramer, F., *"Lines and Parabolas in Taxicab Geometry," Pi Mu Epsilon Journal,* Vol. , No. , 441-448.

2. Reynolds, B., *"Taxicab Geometry," Pi Mu Epsilon Journal,* Vol. 7 , No. 2, 11-88.

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FIGURE 1

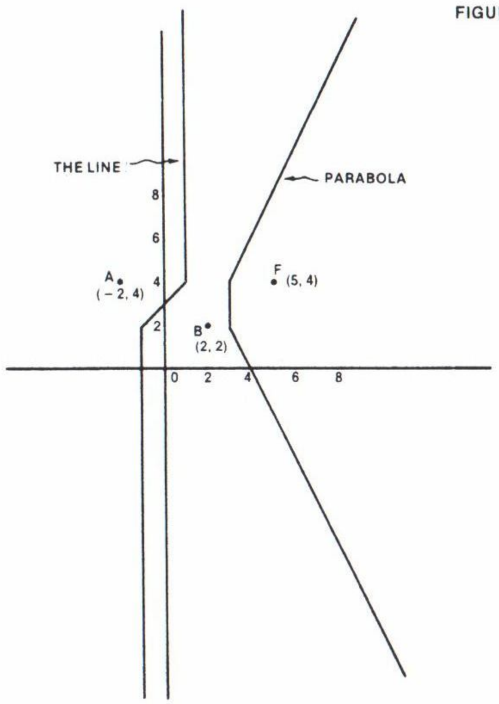


FIGURE 2

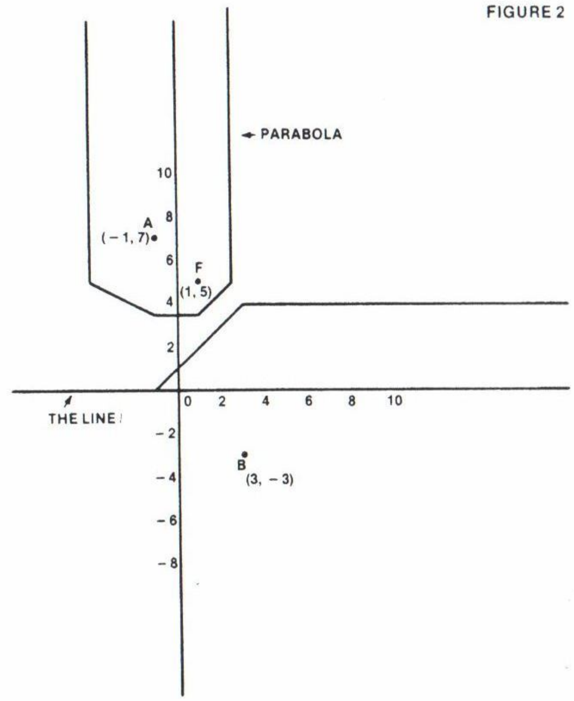


FIGURE 3

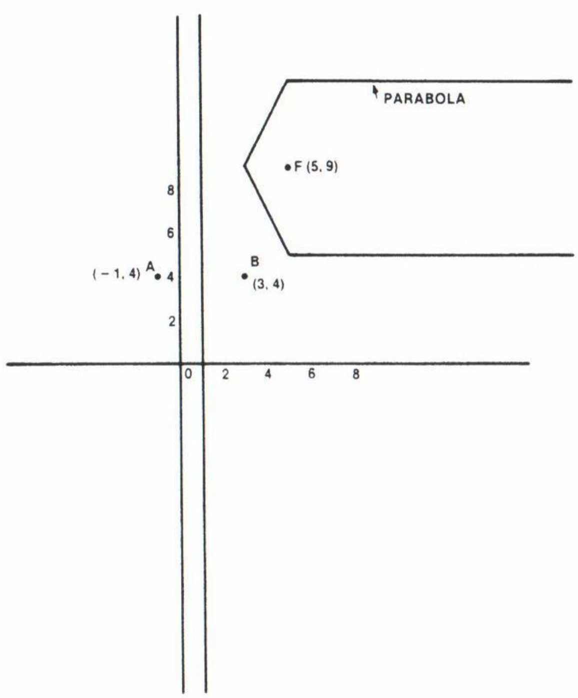


FIGURE 4