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General Equation For Taxicab Conics And Their Classification

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Abstract The taxicab conics introduced and partialy examined by Krause [2]. Laatsch [3], also, introduced some new taxicab conics and established a relationsheep between plane sections of a taxicab cone and these taxicab conics. In this work, a general equation for all taxicab conics is given and they are classified using the coefficients of the general equation as in the Euclidean analogues.

1 Introduction

In [1], K.Menger has introduced the taxicab plane geometry by using the metric

$$d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

instead of the well known Euclidean metric

$$d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for the distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the analytical plane. That is, the *taxicab distance* d_T between points P_1 and P_2 is the lenght of shortest path from P_1 and P_2 composed of line segments parallel to the coordinate axes. In [2] , E.F.Krause developed the taxicab geometry and pointed out while Euclidean geometry appears to be a good model of the natural world, taxicab geometry is a better model of the artificial urban world that man has built.

A few problems related with the taxicab geometry have been studied and developed by some mathematicians. Some of them are Laatsch [3], Sowell [4], Schattschneider [5], Kaya-Akça [6] and Ekici-Kocayusufoğlu-Akça [7].

As it is known there are alternative approachs to obtain conics in the Euclidean geometry. In [2], taxicab ellipses and hyperbolas have been examined, by examples, using two - foci definitions and taxicab parabolas using focus - directrix definition. In [3], Laatsch has defined taxicab conics using focus -

directrix approach and showed that all of these taxicab conics can be obtained by projecting plane sections of a square pyramid (the taxicab analogue of the right circular cone) onto a plane perpendicular to the axis of the pyramid. The set of taxicab ellipses and hyperbolas obtained in [2] and [3] are distinct from each other. That is, *two - foci* and *focus - directrix* definitions give different figures, in this case.

In this work, first, a general equation for all taxicab conics is given. Second, the taxicab conics are classified (types of the conics are determined) in terms of the coefficients of the general equation.

2 A General Equation For Taxicab Conics

The set of all conics in the analytical plane is represented by a quadratic equation of two variables. In this section we give taxicab analogous of this representation.

Lemma : Distance of point $P = (x_0, y_0)$ to the line $l : ax + by + c = 0$ in the taxicab plane is

$$d_T(P, l) = \frac{|ax_0 + by_0 + c|}{\max\{|a|, |b|\}}$$

Proof : In taxicab plane the distance from a point P to the line l is defined as

$$\begin{aligned} d_T(P, l) &= \text{The smallest of all the } d_T(P, X) \text{ where } X \in l \\ &= \min_{X \in l} d_T(P, X) \end{aligned}$$

Since X is to be the intersection point of l with the lines $x = x_0$ or $y = y_0$ we have

$$\begin{aligned} d_T(P, l) &= \begin{cases} \min\left\{\left|\dfrac{ax_0 + by_0 + c}{b}\right|, \left|\dfrac{ax_0 + by_0 + c}{a}\right|\right\} & \text{if } a \neq 0 \neq b \\ \left|\dfrac{by_0 + c}{b}\right| & \text{if } a = 0 \\ \left|\dfrac{ax_0 + c}{a}\right| & \text{if } b = 0 \end{cases} \\ &= (\max\{|a|, |b|\})^{-1} . |ax_0 + by_0 + c| \qquad \square \end{aligned}$$

Theorem 1: Equation of a taxicab conic, with the foci (x_1, y_1) and (x_2, y_2) or with the focus (x_1, y_1) and directrix $ax + by + c = 0$, has the following form

$$|x - x_1| + |y - y_1| + \alpha(|x - x_2| + |y - y_2|) + \beta(|ax + by + c|) \mp \alpha\gamma = 0 \qquad (*)$$

where $\alpha \in \{-1, 0, 1\}$, $\beta = e(\alpha^2 - 1)(\max\{|a|, |b|\})^{-1}$, $\gamma \leq 0$ and e is the eccentricity of related conic.

Proof : Equation of a taxicab conic with the foci (x_1, y_1) and (x_2, y_2) has the form

$$|x - x_1| + |y - y_1| + p(|x - x_2| + |y - y_2|) = \mp q \qquad (1)$$

where $p \in \{-1, 1\}$ and $q \geq 0$. Similarly, equation of a taxicab conic with the focus (x_1, y_1) and directrix $l :\ ax + by + c = 0$ has the form

$$|x - x_1| + |y - y_1| + r\,|ax + by + c| = 0,\ r < 0 \qquad (2)$$

Hence the linear combination of (1) and (2)

$$a_1(|x - x_1| + |y - y_1| + a_2(|x - x_2| + |y - y_2)) + a_3|ax + by + c| \mp a_4 = 0 \quad (3)$$

represent all of the taxicab conics. Where $a_1 \neq 0$, say $a_1 = 1$. Eq.(3) contains Eq.(1) iff $a_2 \in \{-1, 1\}$, $a_3 = 0$, $a_4 \leq 0$. Thus $a_3 = (a_2^2 - 1)s$, $s \in R$. Eq.(3) contains Eq.(2) iff $a_2 = 0$, $a_3 < 0$, and $a_4 = 0$. Consequently, $a_4 = a_2\gamma$ and $s > 0$, $\gamma \leq 0$. Now, let $a_2 = \alpha$. Thus Eq.(3) becomes

$$|x - x_1| + |y - y_1| + \alpha(|x - x_2| + |y - y_2|) + (\alpha^2 - 1)s|ax + by + c| \mp \alpha\gamma = 0 \qquad (4)$$

with $\alpha \in \{-1, 0, 1\}$, $s > 0$ and $\gamma \leq 0$. In the case $\alpha = 0$ we get

$$\frac{|x - x_1| + |y - y_1|}{|ax + by + c|} = s$$

from the above equation. Using Lemma 1, we obtain

$$e = \max\{|a|, |b|\} . \frac{|x - x_1| + |y - y_1|}{|ax + by + c|} = s. \max\{|a|, |b|\}$$

Clearly e represent the eccentricity of the conic. That is,

$$s = e(\max\{|a|, |b|\})^{-1}$$

and

$$(\alpha^2 - 1).s = e.(\alpha^2 - 1).(\max\{|a|, |b|\})^{-1} = \beta$$

which completes the proof. □

Remark

1. Notice that it is not necessary to consider the case where right side of Eq.(1), in the proof is $-q$ when it represents a taxicab ellipse. Therefore, it is sufficient to take $\alpha\gamma$ instead of $\pm\alpha\gamma$ in the general equation when $\alpha = 1$.

2. Clearly, the general equation of taxicab conics can also be given in the form

$$||x - x_1| + |y - y_1| + \alpha(|x - x_2| + |y - y_2|)| + \beta|ax + by + c| + |\alpha|\gamma = 0\,,\; \gamma \leq 0$$

with α and β same as in the theorem.

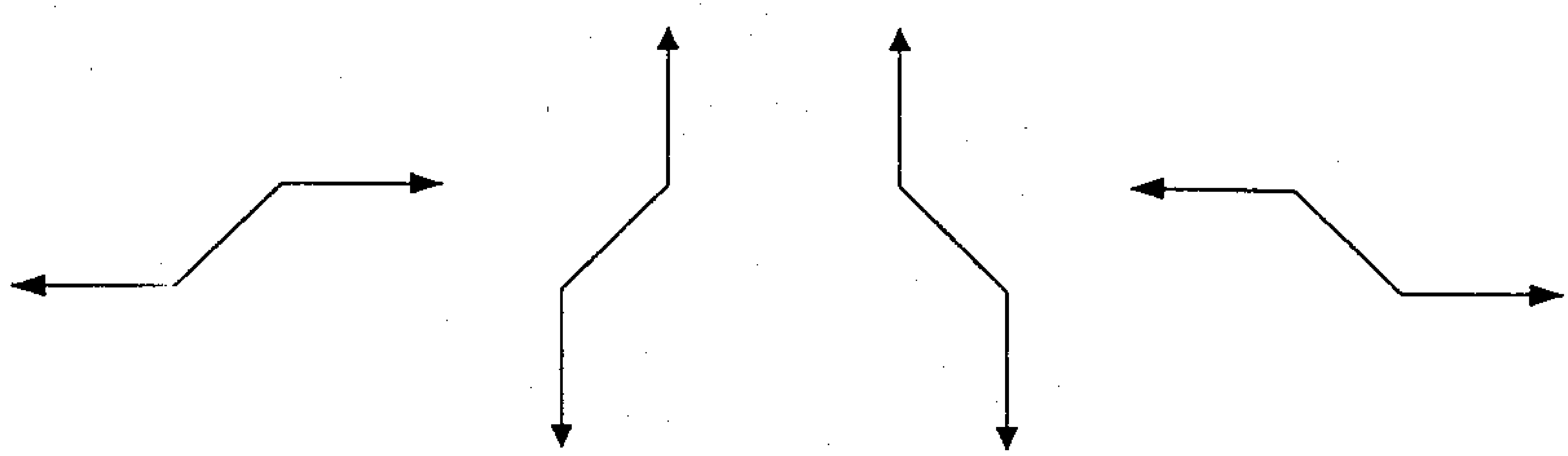
3 Classification of The Taxicab Conics

All of the taxicab conics represent by Eq.(*) are separated into two classes by using the coefficient α. A taxicab conic represented by Eq.(*) is called *a focus-directrix taxicab conic* if $\alpha = 0$ and *a two-foci taxicab conic* if $\alpha = \mp 1$. In particular, it is called *a two-foci taxicab ellipse* if $\alpha = 1$ and *a two-foci taxicab hyperbola* if $\alpha = -1$. Similarly, a focus-directrix conic obtained while $\alpha = 0$ is called a focus-directrix *ellipse, parabola* or *hyperbola* if $0 < e < 1$, $e = 1$ or $e > 1$, respectively. In this section of the paper, we examine types of all these taxicab conics.

First, let us give the abbreviations and concepts that will be used throughout the remaning of the paper : Let $F_1 = (x_1, y_1)$, $F_2 = (x_2, y_2)$ be two fixed points (two-foci) with $x_1 \leq x_2$ and let m denote the slope of the line F_1F_2 in the taxicab plane. $m = \infty$ iff the line F_1F_2 is parallel to $y-$axis.

A line (or a line segment or a ray) is called *horizontal* or *vertical* if it is parallel to $x-$axis or $y-$axis, respectively.

A degenerate line is a plane figure, which consists of a line segment with slope $+1$ or -1 and two horizontal (or two vertical) rays with opposite directions as follows:



The Eq.(*) given in the previous section allows us to classify the taxicab conics as in the following theorems :

Theorem 2: Types of the two-foci taxicab conics represented by Eq.(*) are uniquely determined by the coefficients α, γ and the position of the foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$, as in the following Table 1, Table 2.a, Table 2.b and Table 2.c.

Theorem 3: Types of the focus-directrix taxicab conics represented by Eq.(*) are uniquely determined by the eccentricity e and the position of the directrix $l : ax + by + c = 0$ and the focus $F = F_1 = (x_1, y_1)$ as in the following Table 3.a, Table 3.b and Table 3.c:
Throughout the paper, we let $d_T(F_1, F_2) = \delta$.

	$\|x - x_1\| + \|y - y_1\|$	$+\|x - x_2\| + \|y - y_2\| + \gamma = 0 \quad , \gamma \leq 0$
Position of F_1, F_2, m	γ	Locus
$F_1 = F_2$	$\gamma = 0$	*The point* (x_1, y_1)
$F_1 \neq F_2$		*Empty set*
$F_1 = F_2$	$\gamma < 0$	*The taxicab circle* with center F_1 and radius $\frac{-\gamma}{2}$
$0, \infty$	$-\gamma = \delta$	*The line segment* $[F_1 F_2]$
	$-\gamma > \delta$	*A hexagon*
± 1	$-\gamma = \delta$	*A square region* with diagonal $[F_1 F_2]$
	$-\gamma > \delta$	*An octagon*
$\neq \mp 1$ $\neq 0, \infty$	$-\gamma = \delta$	*A rectangular region* with diagonal $[F_1 F_2]$
	$-\gamma > \delta$	*An octagon*
For all cases	$-\gamma < \delta$	*Empty set*

Table 1. Two-foci Taxicab Ellipses ($\alpha = 1$)

The slope of all line segments which belong to the locus is always an element of $\{-1, 0, 1, \infty\}$ when the locus doesn't consist of a planar region in the above table. If the locus is a planar region then its bordering segments are vertical or horizontal. Taxicab circles, hexagons and octagons can be considered as *true* taxicab ellipses in the family of two-foci taxicab ellipses.

$\|x-x_1\| + \|y-y_1\| - (\|x-x_2\| + \|y-y_2\|) = \pm\gamma \quad , \quad \gamma \le 0$		
Slope of $F_1F_2 = m$	γ	Locus
$0, \infty$	$\gamma = 0$	*The perpendicular bisector* of $[F_1F_2]$
$-1, 1$		*The planar regions connected by a line segment.* $\{(x,y) : (x \le x_1, y \ge y_2)\text{or}(x \ge x_2, y \le y_1)\}$ and the line segment connecting vertices of this regions or $\{(x,y) : (x \le x_1, y \le y_2)\text{or}(x \ge x_2, y \ge y_1)\}$ and the line segment connecting vertices of this regions,according as $m = 1$ or $m = -1$.
$\neq \mp 1$ $\neq 0, \infty$		*A degenerate line*, which consist of a line segment passing through the midpoint of $[F_1F_2]$and two horizontal or two vertical rays starting from the endpoint this line segment,according as $\|m\| > 1$ or $\|m\| < 1$.
$0, \infty$	$-\gamma = \delta$	*The planar region with two sheets.* $R^2 - \{(x,y) : x_1 < x < x_2\}$ or $R^2 \setminus \{(x,y) : \min\{y_1, y_2\} < y < \max\{y_1, y_2\}\}$,according as $x_1 < x_2$ or $x_1 = x_2$. 0
	$-\gamma < \delta$	*Two vertical lines* or *two horizontal lines*, which are perpendicular to the line segment $[F_1F_2]$ such that the distance from the midpoint of $[F_1F_2]$ to these lines is $\frac{-\gamma}{2}$,according as $x_1 < x_2$ or $x_1 = x_2$
$-1, 1$	$-\gamma = \delta$	*The planar regions with two sheets.* $\{(x,y) : (x \le x_1, y \le y_1)$ or $(x \ge x_2, y \ge y_2)\}$ or $\{(x,y) : (x \le x_1, y \ge y_1)$ or $(x \ge x_2, y \le y_2)\}$ with the vertices F_1, F_2, according as $m = 1$ or $m = -1$.
	$-\gamma < \delta$	*A true taxicab hyperbola*, each branch of which consists of a line segment, a vertical ray and a horizontal ray such that the branches are symmetrical with respect to the perpendicular bisector of $[F_1F_2]$.

Table 2.a: Two-foci Taxicab Hyperbolas ($\alpha = -1$)

$\neq \mp 1$ $\neq 0, \infty$	$-\gamma = \delta$	*The planar regions with two sheets.* $\{(x,y) : (x \leq x_1, y \leq y_1) \text{ or } (x \geq x_2, y \geq y_2)\}$ or $\{(x,y) : (x \leq x_1, y \geq y_1) \text{ or } (x \geq x_2, y \leq y_2)\}$ with the vertices F_1, F_2; according as $m > 0$ or $m < 0$.
$(0, 1)$	$-\gamma < \delta$ or $-\gamma = \frac{(1-m)}{(x_2-x_1)^{-1}}$	*Two planar regions each having a tail.* Where the regions are $\{(x,y) : x \leq x_1, y \geq y_2\}$ and $\{(x,y) : x \geq x_2, y \leq y_1\}$ and each tail consists of a line segment and a vertical ray. The sheets of the locus are symmetrical to the midpoint of $[F_1F_2]$.
$(1, \infty)$	$-\gamma < \delta$ or $-\gamma = \frac{(m-1)}{(x_2-x_1)^{-1}}$	*Two planar regions each having a tail.* Where the regions are $\{(x,y) : x \leq x_1, y \geq y_2\}$ and $\{(x,y) : x \geq x_2, y \leq y_1\}$ and each tail consists of a line segment and a horizontal ray. The sheets of the locus are symmetrical to the midpoint of $[F_1F_2]$.
$(-1, 0)$	$-\gamma < \delta$ or $-\gamma = \frac{(1+m)}{(x_2-x_1)^{-1}}$	*Two planar regions each having a tail.* Where the regions are $\{(x,y) : x \leq x_1, y \leq y_2\}$ and $\{(x,y) : x \geq x_2, y \geq y_1\}$ and each tail consists of a line segment and a vertical ray. The sheets of the locus are symmetrical to the midpoint of $[F_1F_2]$.
$(-\infty, -1)$	$-\gamma < \delta$ or $-\gamma = \frac{(1+m)}{(x_1-x_2)^{-1}}$	*Two planar regions each having a tail.* Where the regions are $\{(x,y) : x \leq x_1, y \leq y_2\}$ and $\{(x,y) : x \geq x_2, y \leq y_1\}$ and each tail consists of a line segment and a horizontal ray. The sheets of the locus are symmetrical to the midpoint of $[F_1F_2]$.

Table 2.b (*Continuation* of Table 2.a): Two-foci Taxicab Hyperbolas ($\alpha = -1$)

$(0,1)$	$-\gamma < \delta$ or $-\gamma < \frac{(1-m)}{(x_2-x_1)^{-1}}$	*A pair of parallel degenerate lines.* Where the slope of the line segments are -1 and the rays are vertical.
	$-\gamma < \delta$ or $-\gamma > \frac{(1-m)}{(x_2-x_1)^{-1}}$	*A true taxicab hyperbola.* Where each branch consists of a line segment with slope -1, a vertical ray and a horizontal ray.
$(1,\infty)$	$-\gamma < \delta$ or $-\gamma < \frac{(m-1)}{(x_2-x_1)^{-1}}$	*A pair of parallel degenerate lines.* Where the slope of the line segments are -1 and the rays are horizontal.
	$-\gamma < \delta$ or $-\gamma > \frac{(m-1)}{(x_2-x_1)^{-1}}$	*A true taxicab hyperbola.* Where each branch consists of a line segment with slope -1, a vertical ray and a horizontal ray.
$(-1,0)$	$-\gamma < \delta$ or $-\gamma < \frac{(1+m)}{(x_2-x_1)^{-1}}$	*A pair of parallel degenerate lines.* Where the slope of the line segments are $+1$ and the rays are vertical.
	$-\gamma < \delta$ or $-\gamma > \frac{(1+m)}{(x_2-x_1)^{-1}}$	*A true taxicab hyperbola.* Where each branch consists of a line segment with slope $+1$, a vertical ray and a horizontal ray.
$(-\infty,-1)$	$-\gamma < \delta$ or $-\gamma < \frac{(1+m)}{(x_1-x_2)^{-1}}$	*A pair of parallel degenerate lines.* Where the slope of the line segments are $+1$ and the rays are horizontal.
	$-\gamma < \delta$ or $-\gamma > \frac{(1+m)}{(x_1-x_2)^{-1}}$	*A true taxicab hyperbola.* Where each branch consists of a line segment with slope $+1$, a vertical ray and a horizontal ray.
$\forall m$	$-\gamma > \delta$	*Empty Set*

Table 2.c (*Continuation* of Table 2.b): Two-foci Taxicab Hyperbolas ($\alpha = -1$)

<table>
<tr><th colspan="3">$|x - x_1| + |y - y_1| - e(\max\{|a|, |b|\})^{-1} |ax + by + c| = 0$</th></tr>
<tr><th>e</th><th>$\frac{-a}{b}$</th><th>Locus</th></tr>
<tr><td>$0 < e$
$e < 1$</td><td>$\forall \frac{-a}{b}$</td><td>A quadrilateral, with a vertical diagonal and a horizontal diagonal both passing through F</td></tr>
<tr><td rowspan="2">$e = 1$</td><td>$\left|\frac{-a}{b}\right| = 1$</td><td>A true taxicab parabola, consisting of a line segment, a vertical ray and a horizontal ray. Where the perpendicular bisector of the line segment is axis of symmetry.</td></tr>
<tr><td>$\left|\frac{-a}{b}\right| \neq 1$</td><td>A true taxicab parabola, consisting of two line segments and two horizontal rays or two vertical rays, according as $\left|\frac{-a}{b}\right| > 1$ or $\left|\frac{-a}{b}\right| < 1$</td></tr>
<tr><td rowspan="3">$e > 1$</td><td>$\frac{-a}{b} \in \{0, \infty\}$</td><td>A taxicab hyperbola with two branches.
One of thebranches consists of two line segments and two rays.The second branch consists of only two rays having opposite directions with the first rays. Where the line passing through F and perpendicular to l is axis of symmetry. The four rays are on two lines with the intersection point on the axis of symmetry. The two rays in the second branch start from this point.</td></tr>
<tr><td>$\frac{-a}{b} \in \{-1, 1\}$</td><td>A taxicab hyperbola with two branches.
Each branch consists of a line segment and two rays. Where the line passing through F and perpendicular to l is axis of symmetry. All four rays are on two lines with the intersection point on the axis of symmetry.</td></tr>
<tr><td>$1 \neq \left|\frac{-a}{b}\right| < e$
$a \neq 0$</td><td>A taxicab hyperbola with two branches.
Each branch consists of a line segment and two rays. Where all rays are on two lines with the intersection point on the directrix.</td></tr>
</table>

Table 3.a. Focus-directrix Taxicab Conics ($\alpha = 0$)

$e > 1$	$\left\|\frac{-a}{b}\right\| = e$	A taxicab hyperbola with two branches. One of the branches consists of a line segment and two rays one being vertical. The second branch consists of only two rays one being vertical. The two non-vertical rays are on a line.
	$\left\|\frac{-a}{b}\right\| > e$ $b \neq 0$	A taxicab hyperbola with two branches. One of the branches consists of two line segments and two rays. The second branch consists of only two rays having opposite directions with the first rays. The four rays are on two lines. Each of these two lines, the directrix and a line containing one of the line segments are concurrent.

Table 3.b.(Continuation of Table 3.a) : Focus-directrix Taxicab Conics ($\alpha = 0$)

The focus F is not on the directrix l for the taxicab conics given in Table 3.a.and Table 3.b. A focus-directrix taxicab conic represented by Eq.(*) is called degenerate if the focus is on the directrix.

	$\|x - x_1\| + \|y - y_1\|$	$- e(\max\{\|a\|, \|b\|\})^{-1} \|ax + by + c\| = 0$
e	$-\frac{a}{b}$	Locus
$0 < e < 1$	$\forall(a, b)$	The point $F = (x_1, y_1)$
$e = 1$	$-\frac{a}{b} \in \{-1, 1\}$	The planar regions, $\{(x, y) : (x \leq x_1, y \geq y_1) \text{ or } (x \geq x_1, y \leq y_1)\}$ or $\{(x, y) : (x \leq x_1, y \leq y_1) \text{ or } (x \geq x_1, y \geq y_1)\}$
	$-\frac{a}{b} \notin \{-1, 1\}$	One of the lines $x = x_1$ or $y = y_1$, according as $\|a\| < \|b\|$ or $\|a\| > \|b\|$
$e > 1$	$-\frac{a}{b} = -e$	The line $y = (\frac{1-e}{2})x + \frac{(e+1)by_1 + c(1-e)}{2be}$ which passes through F.
	$-\frac{a}{b} > 1$ and $b \neq 0$	The point $F = (x_1, y_1)$
	$b = 0$ or $(-\frac{a}{b} \leq 1$ and $-\frac{a}{b} \neq -e)$	A pair of lines both passing through F.

Table 3.c Degenerate Focus-directrix Taxicab Conics ($\alpha = 0$)

Proofs : Theorem 2 and Theorem 3 can be easily proven using the restrictions given on the related tables. For instance, suppose that $\alpha = 0$, $e > 1$ and $\left|-\frac{a}{b}\right| > e$. Then Eq.(*) reduces to

$$|x - x_1| + |y - y_1| - e\left|x + ba^{-1}y + ca^{-1}\right| = 0$$

which represents the following four linear equations:

$$l_1 \quad ..... \quad a(1-e)x + (a-eb)y = a(x_1+y_1) + ec$$

if $(x \le x_1, ax+by+c<0, y<y_1)$ or $(x \ge x_1, ax+by+c>0, y>y_1)$; and

$$l_2 \quad ..... \quad a(e-1)x + (a+eb)y = -a(x_1-y_1) - ec$$

if $(x \le x_1, ax+by+c<0, y>y_1)$ or $(x \ge x_1, ax+by+c>0, y<y_1)$; and

$$l_3 \quad ..... \quad a(1+e)x - (a-eb)y = a(x_1-y_1) - ec$$

if $(x > x_1, ax+by+c<0, y \le y_1)$ or $(x < x_1, ax+by+c>0, y \ge y_1)$; and

$$l_4 \quad ..... \quad a(1+e)x + (a+eb)y = a(x_1+y_1) - ec$$

if $(x > x_1, ax+by+c<0, y \ge y_1)$ or $(x < x_1, ax+by+c>0, y \le y_1)$.
Where intersection points of the lines represented by the above equations are
$l_1 \cap l_2 = ((ax_1+bey_1+ec)(a-ae)^{-1}, y_1)$
$l_1 \cap l_3 = (x_1, (aex_1+ay_1+ec)(a-be)^{-1})$
$l_1 \cap_4 = ((bx_1+by_1+c)(b-a)^{-1}, (ax_1+ay_1+c)(a-b)^{-1})$
$l_2 \cap l_3 = ((bx_1-by_1-c)(a+b)^{-1}, (ay_1-ax_1-c)(a+b)^{-1})$
$l_2 \cap l_4 = (x_1, (ay_1-aex_1-ec)(a+be)^{-1})$
$l_3 \cap l_4 = ((ax_1-bey_1-ec)(a+ae)^{-1}, y_1)$.

Now following four subcases are possible :
If $-\frac{a}{b} > e$ and $ax_1+by_1+c>0$ then a simple discussion shows that i) two line segments on the lines l_1 and l_2 in the planar region $(ax_1+bey_1+ec)(a-ae)^{-1} \le x \le x_1$ and ii) the four rays obtained from the lines l_3 and l_4 in the planar regions $x \le (ax_1-bey_1-ec)(a+ae)^{-1}$ or $x \ge x_1$ belong to the locus.
If $-\frac{a}{b} > e$ and $ax_1+by_1+c<0$ then two line segments on the lines l_1 and l_2 in the planar region $x_1 \le x \le (ax_1+bey_1+ec)(a-ae)^{-1}$ and four rays on the lines l_3 and l_4 in the planar regions $x \le x_1$ or $x \ge (ax_1-bey_1-ec)(a+ae)^{-1}$ belong to the locus.
If $-\frac{a}{b} < -e$ and $ax_1+by_1+c<0$ then it is easily seen that the locus consists of two line segments on l_3 and l_4 in the planar region $x_1 \le x \le (ax_1-bey_1-ec)(a+ae)^{-1}$, and four rays on the lines $l_1$and l_2 in the planar regions $x \le x_1$ or $x \ge (ax_1+bey_1+ec)(a-ae)^{-1}$. (See related figure in the Table 4)

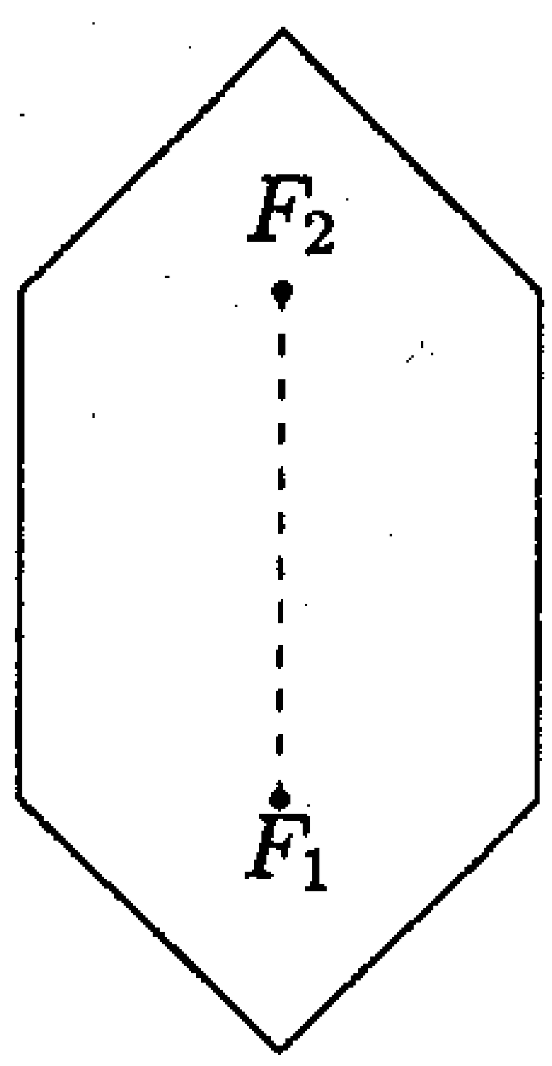
Finally, if $-\frac{a}{b} < -e$ and $ax_1 + by_1 + c > 0$ then the locus consists of two line segments on the lines l_3 and l_4 in the planar region $(ax_1 - bey_1 - ec)(a+ae)^{-1} \leq x \leq x_1$ and four rays on l_1 and l_2 in the planar regions $x \leq (ax_1 + bey_1 + ec)(a - ae)^{-1}$ or $x \geq x_1$.

Furthermore, the points $l_1 \cap l_4$ and $l_2 \cap l_3$ are on the directrix for all cases.

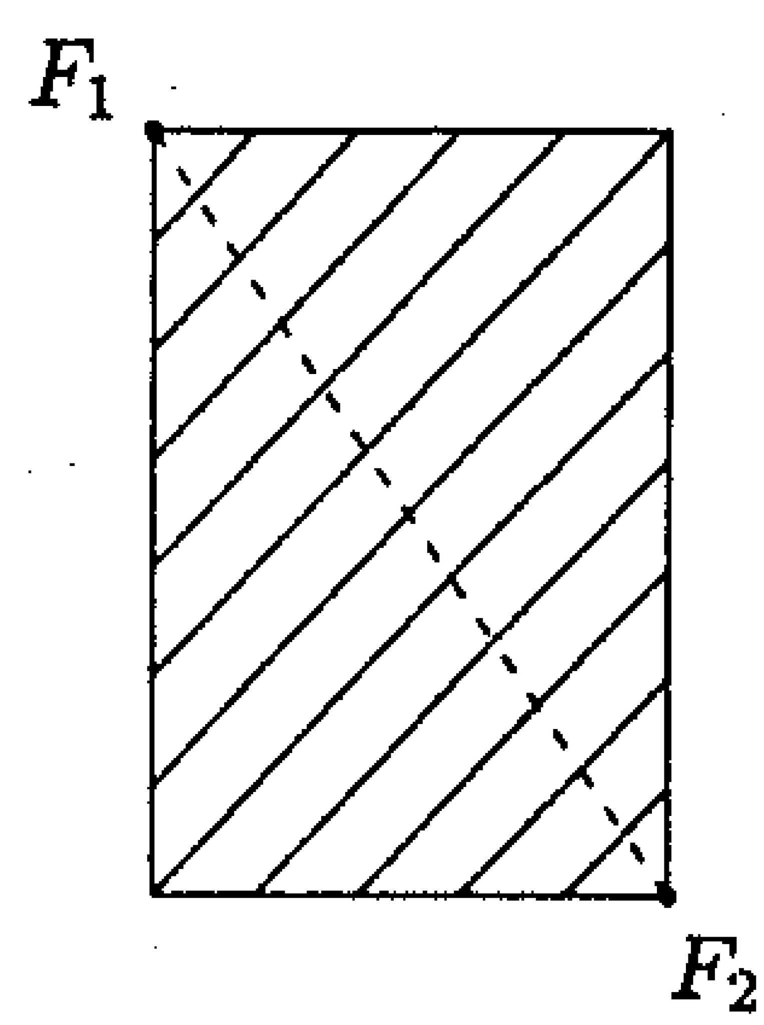
Similar proofs can be given for each of the all remaining cases. □

4 Application: Graphs of The Taxicab Conics

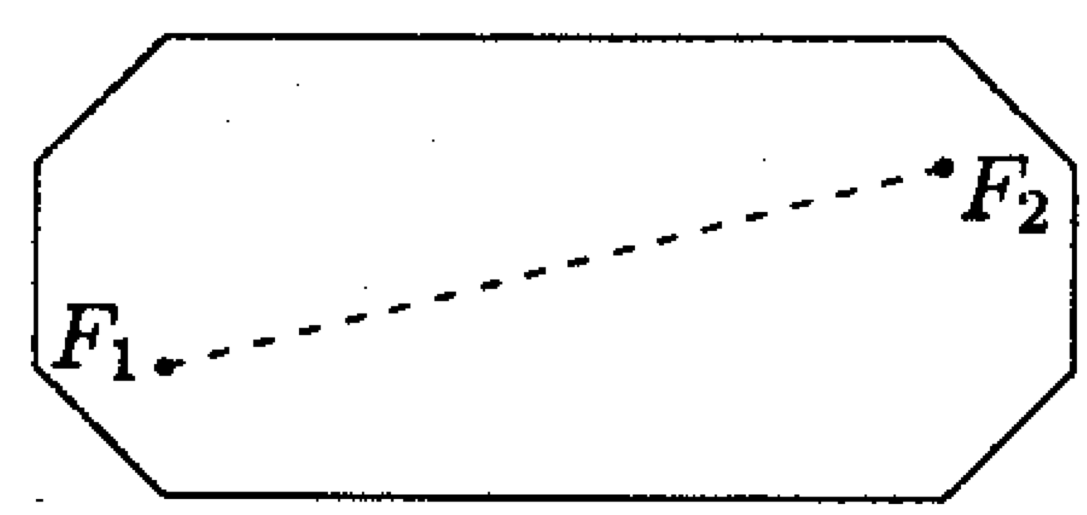
Graph of a taxicab conic can be easily drawn if it is represented by an equation of the form given in theorem 1. Graphs of some of the taxicab conics are given in Table 4.a and Table 4.b.



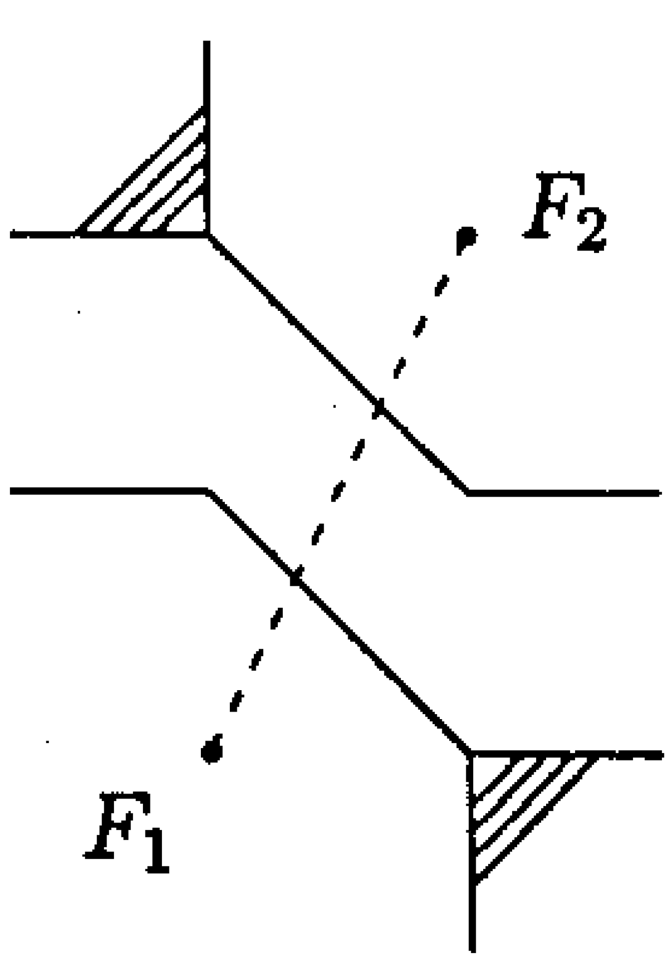
$\alpha = 1, \gamma < 0$
$m = \infty$
$-\gamma \neq d_T(F_1, F_2)$



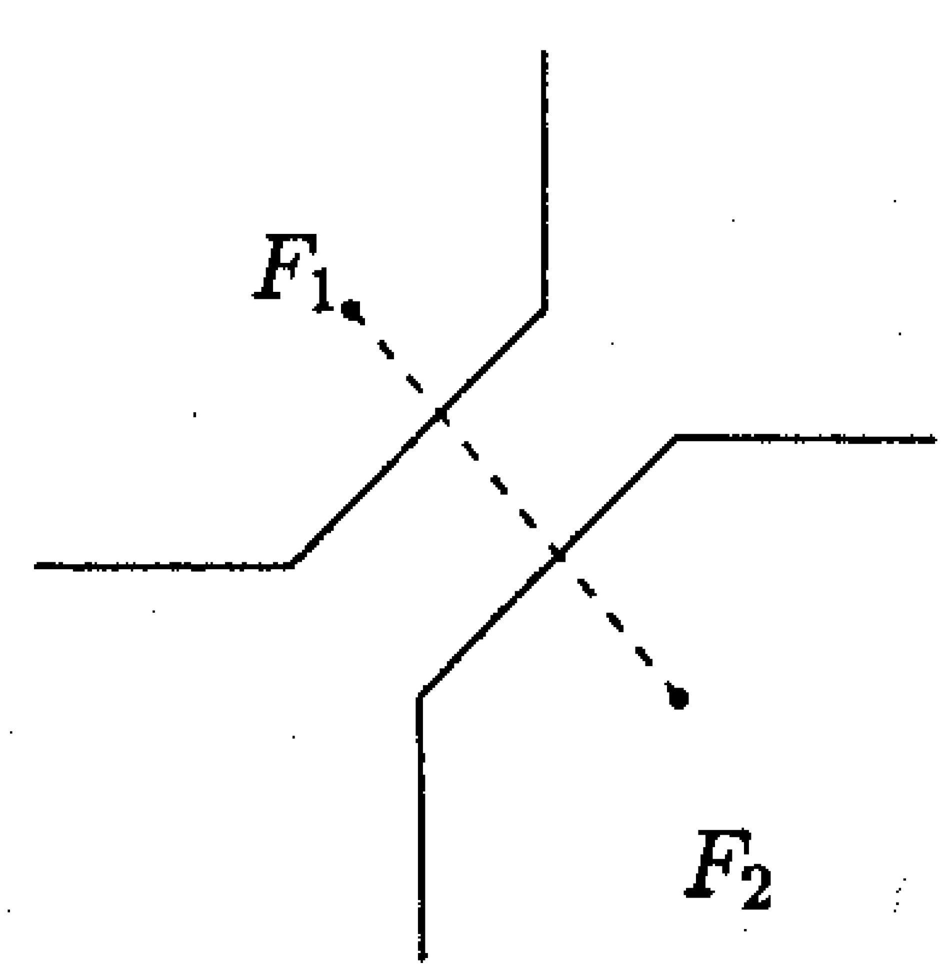
$\alpha = 1, \gamma < 0$
$m \notin \{0, \infty\}$
$-\gamma = d_T(F_1, F_2)$



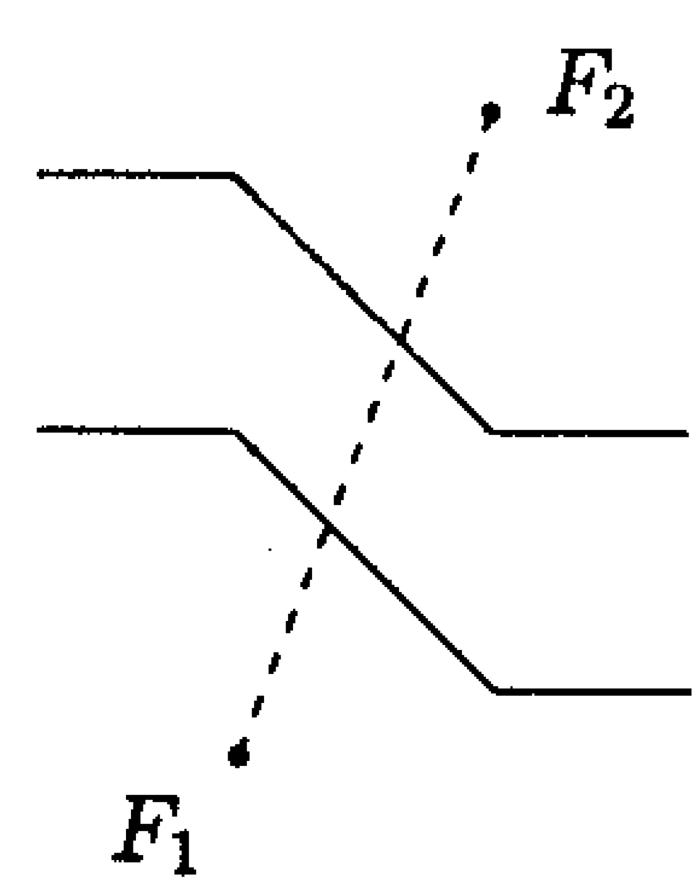
$\alpha = 1, \gamma < 0$
$m \notin \{0, \infty\}$
$-\gamma \neq d_T(F_1, F_2)$



$\alpha = -1, \gamma < 0$
$1 < m \neq \infty$
$-\gamma = x_1 - x_2 + y_2 - y_1$

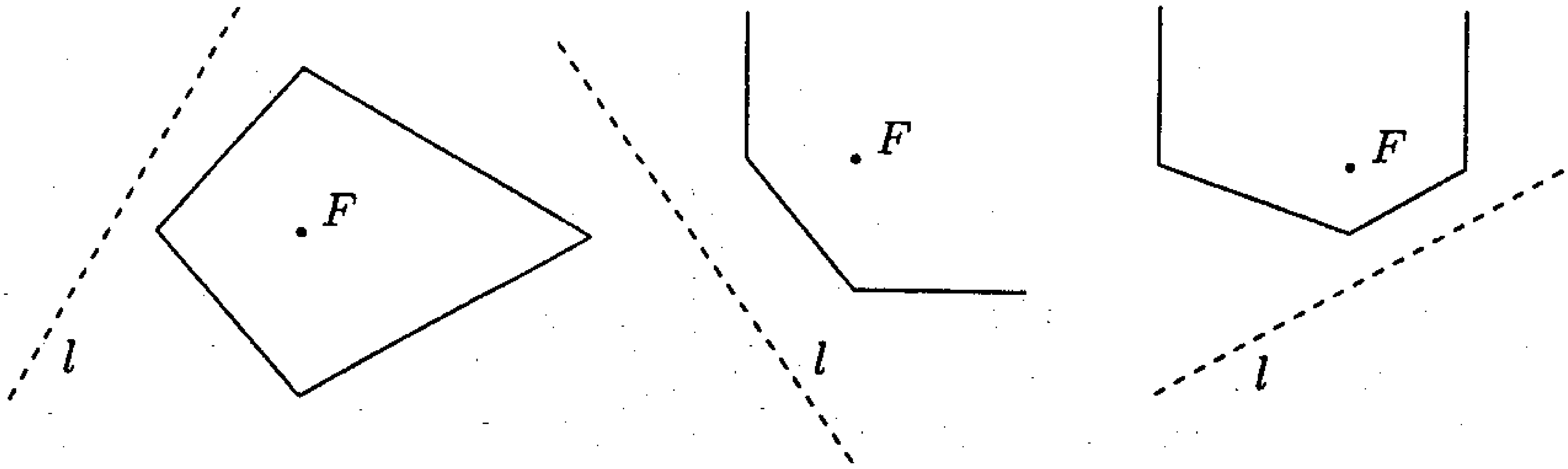


$\alpha = -1, \gamma < 0$
$0 < m \neq \infty$
$-\gamma > |x_2 - x_1 + y_1 - y_2|$
$-\gamma < d_T(F_1, F_2)$

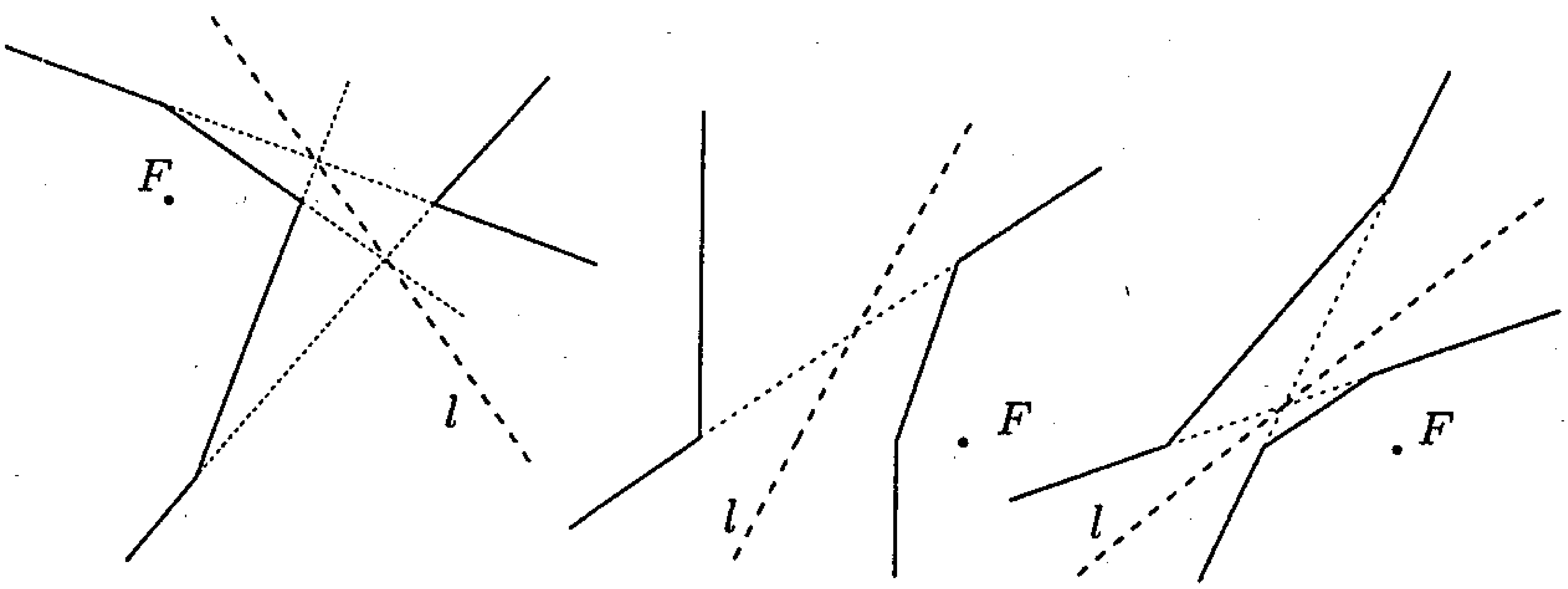


$\alpha = -1, \gamma < 0$
$1 < m \neq \infty$
$-\gamma < |x_2 - x_1 + y_1 - y_2|$
$-\gamma < d_T(F_1, F_2)$

Table 4.a: Graphs of some of the Taxicab Conics



$\alpha = 0, \quad 0 < e < 1$ $\quad$ $\alpha = 0,\ e = 1,\ |-\frac{a}{b}| = 1$ $\quad$ $\alpha = 0,\ e = 1,\ |-\frac{a}{b}| < 1$



$\alpha = 0,\ |-\frac{a}{b}| > e > 1$ $\quad$ $\alpha = 0,\ |-\frac{a}{b}| = e > 1$ $\quad$ $\alpha = 0,\ e > |-\frac{a}{b}| > 1$

Table 4.b : Graphs of some of the Taxicab Conics

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