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THE TAXICAB SPACE GROUP

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Abstract. It is shown that the group of isometries of the 3-dimensional space with respect to taxicab metric is the semi-direct product of octahedral group O_h and $T(3)$, where O_h is the (Euclidean) symmetry group of the regular octahedron and $T(3)$ is the group of all translations of the 3-dimensional space.

1. Introduction

Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). Here the linear structure is the same as the Euclidean one but distance is not *uniform* in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set (Thompson [10] and Jung [3], [4]). Taxicab plane geometry is one of the geometries of this type which has been introduced by Menger [6] and developed by Krause [5]. The plane taxicab geometry has been studied and improved by some mathematicians (for some references see So [9]).

As it is well known, there are three fundamental approaches to geometric study and investigations: synthetic approach, metric approach and group ap-

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proach. The group approach involves isometry groups of a geometry. Then geometry studies those properties which are invariant under the group of motions. In [8], Schattschneider described the group of isometries of the taxicab plane. The three dimensional taxicab space has been analytically studied by Akca–Kaya [1], [2]. In this work we extend the result of Schattschneider to three dimensional taxicab space.

2. Preliminaries

The taxicab 3-dimensional space $\mathbb{R}^3_T$ is almost the same as the Euclidean analytical 3-dimensional space $\mathbb{R}^3$. The points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. The taxicab metric is defined using the distance function

$$d_T(A,B) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

where $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$. According to the definition of d_T-distance the shortest path between the points A and B is the union of three line segments each of which is parallel to a coordinate axis as shown in Fig. 1. Thus, the shortest d_T-distance between A and B is the sum of the Euclidean lengths of such three line segments.

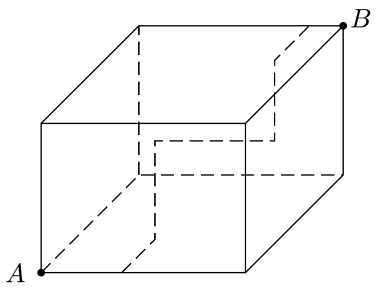

Fig. 1: Some of ways from A to B

The unit ball (unit taxicab sphere) in $\mathbb{R}^3_T$ is the set of points (x, y, z) in the 3-dimensional space which satisfy the equation

$$|x| + |y| + |z| = 1$$

which is an octahedron with vertices $A_1 = (1,0,0)$, $A_2 = (0,1,0)$, $A_3 = (-1,0,0)$, $A_4 = (0,-1,0)$, $A_5 = (0,0,1)$, $A_6 = (0,0,-1)$ as shown in Fig. 2.

We take content of the following lemma from Akca–Kaya [1], which will be useful to determine reflections in $\mathbb{R}^3_T$.

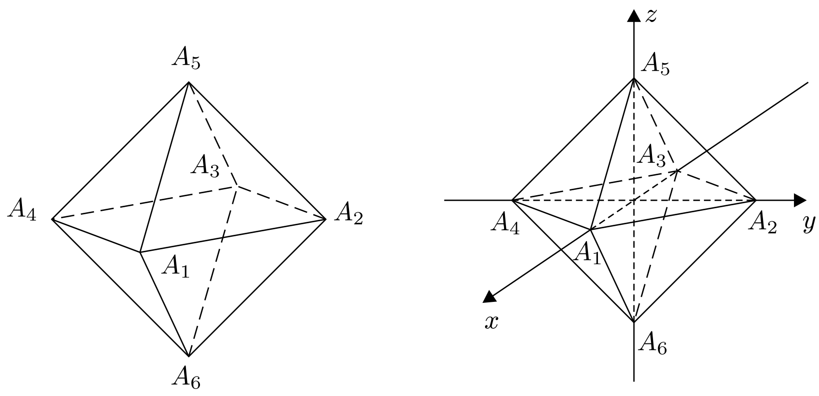

Fig. 2: Graph of the unit taxicab sphere

LEMMA 2.1. *Let l be the line through the points $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$ in the analytical 3-dimensional space and denote d_E the Euclidean metric. If l has direction vector (p,q,r), then*

$$d_T(A,B)=\mu(AB)d_E(A,B), \qquad \mu(AB)=\frac{|p|+|q|+|r|}{\sqrt{p^2+q^2+r^2}}.$$

The above lemma says that d_T-distance along any line is some positive constant multiple of the Euclidean distance along the same line. Thus, one can immediately state the following corollaries:

COROLLARY 2.2. *If A, B and X are any three collinear points in $\mathbb{R}^3$, then $d_E(A,X)=d_E(B,X)$ if and only if $d_T(A,X)=d_T(B,X)$.*

COROLLARY 2.3. *If A, B and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_T(X,A)/d_T(X,B)=d_E(X,A)/d_E(X,B).$$

That is, the ratios of the Euclidean and d_T-distances along a line are the same.

Throughout the article we use the following definitions adopted from Martin [7]: A *transformation* on taxicab 3-space $\mathbb{R}^3_T$ is a one-to-one correspondence from the set of points in the space onto itself. The transformation α is called an *isometry* if $d_T(X,Y)=d_T\big(\alpha(X),\alpha(Y)\big)$ for all points X and Y. The *identity* i is defined by $i(X)=X$ for every point X. The isometry α is a *symmetry* for a set of points if α fixes that set of points. If Δ is a plane, then the *reflection* σ_Δ is the mapping on the points in $\mathbb{R}^3_T$ such that $\sigma_\Delta(X)=X$ if X is on Δ and $\sigma_\Delta(X)=Y$ if X is off Δ and the plane Δ is the perpendicular bisector of the line segment XY. If two planes Γ and Δ intersect at a line l, then $\sigma_\Delta\sigma_\Gamma$ is called a *rotation* about the axis l. If Γ and Δ are two intersecting planes each perpendicular to plane Π, then $\sigma_\Pi\sigma_\Delta\sigma_\Gamma$ is called a *rotary reflection* (a *rotatory reflection*) about the point common to Γ, Δ and Π. If M is a point, the *inversion* σ_M about M is the transformation such that $\sigma_M(X) = Y$ for all points X where M is the midpoint of X and Y. σ_M is sometimes called a *point reflection.* If Γ and Δ are two intersecting planes and M is a point common to Γ, Δ, then $\sigma_M\sigma_\Delta\sigma_\Gamma$ is called a *rotary inversion.* If the planes Γ and Δ are parallel, that is, $\Gamma \parallel \Delta$, then $\sigma_\Delta\sigma_\Gamma$ is a *translation* along the common perpendicular lines to the planes Γ and Δ.

In the remaining part of this work, we will study the isometries of $\mathbb{R}^3_T$, and determine its group of isometries.

3. Isometries of the taxicab space ($\mathbb{R}^3_T$)

One of the basic problems in geometric investigations for a given space S with a metric d is to describe the group G of isometries. If S is the Euclidean 3-dimensional space with the usual metric, then it is well known that G consists of all translations, rotations, reflections, glide reflections, rotary reflection and screw of the 3-dimensional space.

It is known that for the Euclidean 3-dimensional space $G = E(3)$ is the semi-direct product of its two subgroups $O(3)$ (the orthogonal group) and $T(3)$, where $O(3)$ is the symmetry group of the unit sphere and $T(2)$ (the translation group) consists of all translations of the 3-dimensional space.

Since the taxicab space geometry is the study of Euclidean points, lines, planes and angles in $\mathbb{R}^3$, an isometry of $\mathbb{R}^3_T$ is therefore an isometry of the real space with respect to the d_T metric.

Proposition 3.1. *Every Euclidean translation of $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3_T$.*

Proof. Let $T_A : \mathbb{R}^3_T \to \mathbb{R}^3_T$ such that $T_A(X) = A + X$ is a translation as in the real 3-dimensional space $\mathbb{R}^3$, where $A = (a_1, a_2, a_3)$ and $X = (x_1, y_1, z_1) \in \mathbb{R}^3_T$. For $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2) \in \mathbb{R}^3_T$, we have

$$d_T\big(T_A(X), T_A(Y)\big) = \big|(a_1 + x_1) - (a_1 + x_2)\big| + \big|(a_2 + y_1) - (a_2 + y_2)\big|$$

$$+ \big|(a_3 + z_1) - (a_3 + z_2)\big| = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2| = d_T(X, Y).$$

That is, T_A is an isometry. □

Notice that it is enough to consider the planes passing through the origin to find the rotations and the reflections in $\mathbb{R}^3_T$ by Proposition 3.1. The following lemma helps to determine the reflections which preserve the distance in $\mathbb{R}^3_T$.

LEMMA 3.2. *A reflection about the plane* $\Delta: ax+by+cz=0$ *in* $\mathbb{R}^3_T$ *is an isometry if and only if* (a,b,c) *is parallel to an element of the set of vectors*

$$D=\big\{(1,0,0),(0,1,0),(0,0,1),(\pm1,1,0),(1,0,\pm1),(0,1,\pm1)\big\}.$$

PROOF. Without loss of generality consider the Euclidean reflection σ_Δ about the plane $ax+by+cz=0$ with the unit normal vector (a,b,c), $\sigma_\Delta:\mathbb{R}^3_T\to\mathbb{R}^3_T$ such that

$$\sigma_\Delta(x,y,z)=\big((1-2a^2)x-2aby-2acz,-2abx+(1-2b^2)y-2bcz,$$
$$-2acx-2bcy+(1-2c^2)z\big).$$

Since the vector set $\big\{E_1=(1,0,0),E_2=(0,1,0),E_3=(0,0,1)\big\}$ is a base of $\mathbb{R}^3$, a reflection which preserves this base must be an isometry. So it is sufficient to determine the reflections which preserve vectors of this base. Applying Euclidean reflection to the vectors of the base, one can get

$$\sigma_\Delta(1,0,0)=(1-2a^2,-2ab,-2ac),$$
$$\sigma_\Delta(0,1,0)=(-2ab,1-2b^2,-2bc),$$
$$\sigma_\Delta(0,0,1)=(-2ac,-2bc,1-2c^2).$$

Clearly $d_T(O,E_1)=d_T(O,E_2)=d_T(O,E_3)=1$. If a reflection preserves d_T-distance, we must look for a, b, c which implies

$$d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_1)\big)=d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_2)\big)=d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_3)\big)=1.$$

Thus,

$$\begin{array}{l} d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_1)\big)=1\\ d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_2)\big)=1\\ d_T\big(\sigma_\Delta(O),\sigma_\Delta(E_3)\big)=1\end{array}\quad\Leftrightarrow\quad\begin{cases}|2a^2-1|+|2ab|+|2ac|=1\\ |2ab|+|2b^2-1|+|2bc|=1\\ |2ac|+|2bc|+|2c^2-1|=1.\end{cases}$$

Now, one can solve this system of equations and obtain the solutions

$$(1,0,0),\ (0,1,0),\ (0,0,1),$$
$$\left(\pm\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right),\ \left(\frac{1}{\sqrt2},0,\pm\frac{1}{\sqrt2}\right)\quad\text{and}\quad\left(0,\frac{1}{\sqrt2},\pm\frac{1}{\sqrt2}\right).$$

Conversely, if (p_1, q_1, r_1) and (p_2, q_2, r_2) are the direction vectors of the lines OX and OY, where $\sigma_\Delta(X) = Y$, then

$$d_E(O,X) = d_E(O,Y), \quad d_T(O,X) = \mu(OX)d_E(O,X),$$
$$d_T(O,Y) = \mu(OY)d_E(O,Y) = \mu(OY)d_E(O,X),$$

where

$$\mu(AB) = \frac{|p| + |q| + |r|}{\sqrt{p^2 + q^2 + r^2}}$$

by Lemma 2.1. Now it is easy to check that $\mu(OX) = \mu(OY)$ for all possible cases in the following table, which implies $d_T(O,X) = d_T(O,Y)$.

Δ	(p_2, q_2, r_2)
$x = 0$	$(-p_1, q_1, r_1)$
$y = 0$	$(p_1, -q_1, r_1)$
$z = 0$	$(p_1, q_1, -r_1)$
$x + y = 0$	$(-2q_1, -2p_1, 2r_1)$
$x - y = 0$	$(2q_1, 2p_1, 2r_1)$

Δ	(p_2, q_2, r_2)
$x + z = 0$	$(-2r_1, 2q_1, -2p_1)$
$x - z = 0$	$(2r_1, 2q_1, 2p_1)$
$y + z = 0$	$(2p_1, -2r_1, -2q_1)$
$y - z = 0$	$(2p_1, 2r_1, 2q_1)$

□

COROLLARY 3.3. *The set S_T of isometric reflections about the planes passing through the origin consists of the nine Euclidean reflections about the planes with the equations $x = 0$, $y = 0$, $z = 0$, $x + y = 0$, $x + z = 0$, $y + z = 0$, $x - y = 0$, $x - z = 0$, $y - z = 0$.*

LEMMA 3.4. *A rotation r_θ about a line l passing through $O = (0,0,0)$ is an isometry if and only if $r_\theta \in R_T = R_1 \cup R_2 \cup R_3$ such that*

$$R_1 = \left\{ r_\theta \mid \theta \in \{\pi/2, \pi, 3\pi/2\},\ \textit{rotation axis has a direction vector in } D_1 \right\},$$

$$R_2 = \left\{ r_\theta \mid \theta \in \{2\pi/3, 4\pi/3\},\ \textit{rotation axis has a direction vector in } D_2 \right\},$$

$$R_3 = \{ r_\theta \mid \theta = \pi,\ \textit{rotation axis has a direction vector in } D_3 \}$$

where

$$D_1 = \left\{ (1,0,0), (0,1,0), (0,0,1) \right\},$$

$$D_2 = \left\{ (1,1,1), (-1,1,1), (1,-1,1), (1,1,-1) \right\}$$

and

$$D_3 = \left\{ (1,1,0), (1,0,1), (0,1,1), (1,-1,0), (1,0,-1), (0,1,-1) \right\}.$$

PROOF. Consider the Euclidean rotation r_θ about a line l with the direction of a unit vector (p,q,r), $r_\theta:\mathbb{R}^3_T\to\mathbb{R}^3_T$ which has a matrix representation as follows:

$$\begin{bmatrix} \cos\theta+p^2(1-\cos\theta) & pq(1-\cos\theta)-r\sin\theta & pr(1-\cos\theta)+q\sin\theta \\ pq(1-\cos\theta)+r\sin\theta & \cos\theta+q^2(1-\cos\theta) & qr(1-\cos\theta)-p\sin\theta \\ pr(1-\cos\theta)-q\sin\theta & qr(1-\cos\theta)+p\sin\theta & \cos\theta+r^2(1-\cos\theta) \end{bmatrix}.$$

Since a rotation about an axis l can be expressed as composition of two reflections by two planes intersecting along the line l, it is sufficient to consider the rotations about the lines with direction vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,-1,0)$, $(1,0,-1)$, $(0,1,-1)$, $(1,1,1)$, $(-1,1,1)$, $(1,-1,1)$ or $(1,1,-1)$ by Corollary 3.3. In order to find isometric rotations in $\mathbb{R}^3_T$, it will be enough to determine the rotations which preserve the lengths of the edges of the d_T-unit sphere.

Now consider the vertices $A_1=(1,0,0)$ and $A_2=(0,1,0)$ of the d_T-unit sphere. Rotating A_1 and A_2 by an angle θ about the line l, we get

$$r_\theta(A_1)=\big(\cos\theta+p^2(1-\cos\theta),pq(1-\cos\theta)+r\sin\theta,$$
$$pr(1-\cos\theta)-q\sin\theta\big)$$
$$r_\theta(A_2)=\big(pq(1-\cos\theta)-r\sin\theta,\cos\theta+q^2(1-\cos\theta),$$
$$qr(1-\cos\theta)+p\sin\theta\big).$$

Clearly $d_T(A_1,A_2)=2$. If r_θ preserves d_T-distances, we must look for θ, $\theta\neq 0$, which implies $d_T\big(r_\theta(A_1),r_\theta(A_2)\big)=2$. Thus,

$$d_T\big(r_\theta(A_1),r_\theta(A_2)\big)=\big|\cos\theta+p^2(1-\cos\theta)-pq(1-\cos\theta)-r\sin\theta\big|$$
$$+\big|pq(1-\cos\theta)+r\sin\theta-\cos\theta+q^2(1-\cos\theta)\big|$$
$$+\big|pr(1-\cos\theta)-q\sin\theta-qr(1-\cos\theta)+p\sin\theta\big|=2.$$

If the direction vector of l is in D_1, say $(1,0,0)$, then $(p,q,r)=(1,0,0)$. Using these values of p, q, r in the equation $d_T\big(r_\theta(A_1),r_\theta(A_2)\big)=2$ one obtains

$$|-\cos\theta|+|\sin\theta|=1.$$

From this equation, one gets $\theta=\pi/2$, π or $3\pi/2$. That is, every Euclidean rotation about the x-axis with $\theta=\pi/2$, π or $3\pi/2$ is an isometry of $\mathbb{R}^3_T$. Similarly, if the direction vector of l is one of $(0,1,0)$ or $(0,0,1)$, then $\theta=\pi/2$, π or $3\pi/2$.

If the direction vector of l is in D_2, say $(1,1,1)$, then $(p,q,r) = \big(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\,\big)$. Using these values of p,q,r in the equation $d_T\big(r_\theta(A_1), r_\theta(A_2)\big) = 2$ one gets

$$\big|\sqrt{3}\cos\theta + \sin\theta\,\big| + \big|\sqrt{3}\cos\theta - \sin\theta\,\big| + 2|\sin\theta| = 2\sqrt{3}.$$

From this equation, one obtains $\theta = 2\pi/3$ or $4\pi/3$. That is, every Euclidean rotation about the line l that has the direction vector $(1,1,1)$ with $\theta = 2\pi/3$ or $4\pi/3$ is an isometry of $\mathbb{R}^3_T$. Similarly, if the direction vector of l is one of $(-1,1,1)$, $(1,-1,1)$, or $(1,1,-1)$, then $\theta = 2\pi/3$, or $4\pi/3$.

If the direction vector of l is in D_3, say $(1,1,0)$, then $(p,q,r) = \big(1/\sqrt{2}, 1/\sqrt{2}, 0\big)$. Using these values of p, q, r in the equation $d_T\big(r_\theta(A_1), r_\theta(A_2)\big) = 2$ one gets

$$|\cos\theta| + |\sin\theta|/\sqrt{2} = 1.$$

From this equation, one gets $\theta = \pi$. That is, every Euclidean rotation about the line l that has the direction vector $(1,1,0)$ with $\theta = \pi$ is an isometry of $\mathbb{R}^3_T$. Similarly, if the direction vector of l is one of $(1,0,1)$, $(0,1,1)$, $(1,-1,0)$, $(1,0,-1)$ or $(0,1,-1)$, then $\theta = \pi$.

Conversely, if $r_\theta(X) = Y$, then it can be easily checked that $\mu(OX) = \mu(OY)$ for all possible cases as in Lemma 3.2. For instance:

rotation	$(1,0,0)$ $\theta = \pi/2$	$\frac{1}{\sqrt{2}}(1,1,0)$ $\theta = \pi$	$\frac{1}{\sqrt{3}}(1,1,1)$ $\theta = 2\pi/3$	$\dots$
(p_2,q_2,r_2)	$(p_1,-r_1,q_1)$	$(q_1,p_1,-r_1)$	(r_1,p_1,q_1)	$\dots$

□

Clearly the *inversion* σ_O about $O = (0,0,0)$, which maps (x,y,z) to $(-x,-y,-z)$, is an isometry of $\mathbb{R}^3_T$ and preserves the d_T-unit sphere. σ_O will be useful to give the following two lemmas:

COROLLARY 3.5. *The set R_T of isometric rotations about the lines passing through the origin consists of exactly twenty-three Euclidean rotations.*

LEMMA 3.6. *There are only six rotary reflections about O that preserve the d_T-distances.*

PROOF. Since a rotary reflection $\rho := \sigma_\Pi\sigma_\Delta\sigma_\Gamma = \sigma_\Pi r_\theta$, $r_\theta \in R_T$, Γ and Δ perpendicular to Π, one has to consider all the possible 9 cases for Π and 13 choices for the axes of rotations by Corollary 3.3 and Lemma 3.4. But notice that none of the lines in D_2 is perpendicular to any one of the planes given in Corollary 6.

If Π stands for the plane $x = 0$, then $(1,0,0)$ is unit direction vector of r_θ and

$$\rho(x,y,z) = \sigma_\Pi r_\theta(x,y,z) = (-x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta).$$

Consequently, $\rho(A_1)=(-1,0,0)$, $\rho(A_2)=(0,\cos\theta,\sin\theta)$. Since $d_T(A_1,A_2) = 2$,

$$d_T\big(\rho(A_1),\rho(A_2)\big) = 2 \Leftrightarrow |\cos\theta|+|\sin\theta|=1 \Leftrightarrow \theta\in\{\pi/2,\pi,3\pi/2\}.$$

Now it is easy to show that $\sigma_\Pi r_\pi=\sigma_O$, and therefore there are exactly two rotary reflections obtained using the plane $x=0$. Similarly, one can easily obtain new rotary reflections using the planes $y=0$ and $z=0$ as Π.

If Π denotes the plane $x+y=0$, then $\big(1/\sqrt{2},1/\sqrt{2},0\big)$ is unit direction vector of r_θ and

$$\rho(x,y,z)=\sigma_\Pi r_\theta(x,y,z)=\Bigg(\left(\frac{\cos\theta-1}{2}\right)x-\left(\frac{1+\cos\theta}{2}\right)y+\frac{\sin\theta}{\sqrt{2}}z,$$

$$\left(\frac{-1-\cos\theta}{2}\right)x+\left(\frac{\cos\theta-1}{2}\right)y-\frac{\sin\theta}{\sqrt{2}}z,\frac{-\sin\theta}{\sqrt{2}}x+\frac{\sin\theta}{\sqrt{2}}y+\cos\theta z\Bigg).$$

Clearly

$$\rho(A_1)=\left(\frac{1}{2}(\cos\theta-1),\frac{-1}{2}(1+\cos\theta),\frac{-1}{\sqrt{2}}\sin\theta\right),$$

$$\rho(A_2)=\left(\frac{-1}{2}(1+\cos\theta),\frac{1}{2}(\cos\theta-1),\frac{1}{\sqrt{2}}\sin\theta\right)$$

and

$$d_T\big(\rho(A_1),\rho(A_2)\big) = 2 \Leftrightarrow \sqrt{2}|\cos\theta|+|\sin\theta|=2 \Leftrightarrow \theta=\pi$$

but $\rho=\sigma_\Pi r_\pi=\sigma_O$ the inversion about $O=(0,0,0)$. That is, there is no new rotary reflection in this case.

Similarly, it is easily seen that there is no new rotary reflection if Π is any of the remaining planes $x-y=0$, $x+z=0$, $x-z=0$, $y+z=0$, $y-z=0$ and the rotations axes are parallel to any one of $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$, respectively. □

Lemma 3.7. *There exist only eight rotary inversions about O that preserve the d_T-distances.*

Proof. Since a rotary inversion $\rho:=\sigma_O\sigma_\Delta\sigma_\Gamma=\sigma_O r_\theta$, $r_\theta\in R_T$, one has to consider 13 possible cases for the axes of rotations by Lemma 3.4.

If r_θ denotes the rotations about the x-axis, then $(1,0,0)$ is the unit direction vector of r_θ and $\rho(x,y,z)=\sigma_O r_\theta(x,y,z)=(-x,-y\cos\theta+z\sin\theta,$ $-y\sin\theta-z\cos\theta)$. Consequently, $\rho(A_1)=(-1,0,0)$, $\rho(A_2)=(0,-\cos\theta,$ $-\sin\theta)$. Since $d_T(A_1,A_2)=2$,

$$d_T\big(\rho(A_1),\rho(A_2)\big) = 2 \Leftrightarrow |\cos\theta|+|\sin\theta|=1 \Leftrightarrow \theta\in\{\pi/2,\pi,3\pi/2\}.$$

Now it is easy to show that $\sigma_O r_\theta$ is a rotary reflection or reflection, and therefore there is no new rotary inversion in this case. Similarly, one can easily see that there is no new rotary inversion using the rotations about the y, z-axis.

If r_θ stands for the rotation about l parallel to $(1,1,0)$, then $\big(1/\sqrt{2},1/\sqrt{2},0\big)$ is unit direction vector of r_θ and

$$\rho(x,y,z)=\sigma_O r_\theta(x,y,z)=\bigg(\Big(\frac{-1-\cos\theta}{2}\Big)x+\Big(\frac{\cos\theta-1}{2}\Big)y-\frac{\sin\theta}{\sqrt{2}}z,$$

$$\Big(\frac{\cos\theta-1}{2}\Big)x+\Big(\frac{-1-\cos\theta}{2}\Big)y+\frac{\sin\theta}{\sqrt{2}}z,\frac{\sin\theta}{\sqrt{2}}x-\frac{\sin\theta}{\sqrt{2}}y-\cos\theta z\bigg).$$

Obviously

$$\rho(A_1)=\Big(\frac{-1}{2}(1+\cos\theta),\frac{1}{2}(\cos\theta-1),\frac{1}{\sqrt{2}}\sin\theta\Big),$$

$$\rho(A_2)=\Big(\frac{1}{2}(\cos\theta-1),\frac{-1}{2}(1+\cos\theta),\frac{-1}{\sqrt{2}}\sin\theta\Big)$$

and

$$d_T\big(\rho(A_1),\rho(A_2)\big)=2\Leftrightarrow\sqrt{2}|\cos\theta|+|\sin\theta|=2\Leftrightarrow\theta=\pi$$

but $\rho=\sigma_O r_\pi$ is a rotary reflection or reflection. That is, there is no new rotary inversion in this case.

Similarly, it is easily seen that there is no new rotary inversion if r_θ is any of the remaining rotation axes parallel to $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$.

If r_θ denotes rotation about an axis parallel to $(1,1,1)$, then $\big(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}\,\big)$ is the unit direction vector of r_θ and

$$\rho(x,y,z)=\sigma_O r_\theta(x,y,z)=\bigg(\Big(\frac{-2\cos\theta-1}{3}\Big)x$$

$$+\Big(\frac{\cos\theta+\sqrt{3}\sin\theta-1}{3}\Big)y+\Big(\frac{\cos\theta-\sqrt{3}\sin\theta-1}{3}\Big)z,$$

$$\Big(\frac{\cos\theta-\sqrt{3}\sin\theta-1}{3}\Big)x+\Big(\frac{-2\cos\theta-1}{3}\Big)y+\Big(\frac{\cos\theta+\sqrt{3}\sin\theta-1}{3}\Big)z,$$

$$\left(\left(\frac{\cos\theta+\sqrt{3}\sin\theta-1}{3}\right)x+\left(\frac{\cos\theta-\sqrt{3}\sin\theta-1}{3}\right)y+\left(\frac{-2\cos\theta-1}{3}\right)z\right).$$

Clearly

$$\rho(A_1)=\left(\frac{-2\cos\theta-1}{3},\frac{\cos\theta-\sqrt{3}\sin\theta-1}{3},\frac{\cos\theta+\sqrt{3}\sin\theta-1}{3}\right),$$

$$\rho(A_2)=\left(\frac{\cos\theta+\sqrt{3}\sin\theta-1}{3},\frac{-2\cos\theta-1}{3},\frac{\cos\theta-\sqrt{3}\sin\theta-1}{3}\right)$$

and

$$d_T\big(\rho(A_1),\rho(A_2)\big)=2\ \Leftrightarrow\ \left|3\cos\theta+\sqrt{3}\sin\theta\right|+\left|3\cos\theta-\sqrt{3}\sin\theta\right|$$
$$+\left|2\sqrt{3}\sin\theta\right|=6\ \Leftrightarrow\ \theta\in\{2\pi/3,4\pi/3\}.$$

Therefore there are exactly two rotary inversions obtained using rotation about the axis parallel to $(1,1,1)$.

Similarly, it is easily obtained that there are two new rotary inversions each of the remaining rotation axes parallel to $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$. That is, there are eight rotary inversions that preserve d_T-distance. □

It can be easily checked that $\sigma_O\sigma_\Delta=r_\pi$, $r_\pi\in R_1\cup R_3$. Thus we have the *octahedral group* O_h, consisting of nine reflections about planes, twenty-three rotations, six rotary reflections, eight rotary inversions, one inversion and the identity. That is, the Euclidean symmetry group of the regular octahedron.

Now, let us show that all isometries of $\mathbb{R}^3_T$ are in $T(3).O_h$.

Definition 3.8. Let $A=(a_1,a_2,a_3)$, $B=(b_1,b_2,b_3)$ be two points in $\mathbb{R}^3_T$. The *minimum distance set* of A, B is defined by

$$\big\{X\mid d_T(A,X)+d_T(B,X)=d_T(A,B)\big\}$$

and is denoted by $[AB]$.

In general, $[AB]$ represents a rectangular prism with diagonal AB as in Fig. 3.

In particular, if AB is parallel to any of the coordinate axes, then $[AB]=AB$.

Proposition 3.9. *Let* $\phi:\mathbb{R}^3_T\to\mathbb{R}^3_T$ *be an isometry and let* $[AB]$ *be the rectangular prism. Then*

$$\phi\big([AB]\big)=\big[\phi(A)\phi(B)\big].$$

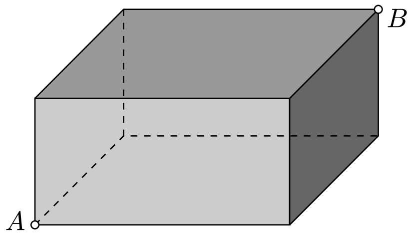

Fig. 3

PROOF. Let $Y \in \phi\big([AB]\big)$. Then,

$$Y \in \phi\big([AB]\big) \;\Leftrightarrow\; \exists X \in [AB] \ni Y = \phi(X)$$
$$\Leftrightarrow\; d_T(A,X) + d_T(X,B) = d_T(A,B)$$
$$\Leftrightarrow\; d_T\big(\phi(A),\phi(X)\big) + d_T\big(\phi(X),\phi(B)\big) = d_T\big(\phi(A),\phi(B)\big)$$
$$\Leftrightarrow\; Y = \phi(X) \in \big[\phi(A)\phi(B)\big]. \quad \square$$

COROLLARY 3.10. *Let $\phi : \mathbb{R}^3_T \to \mathbb{R}^3_T$ be an isometry and let $[AB]$ be a rectangular prism. Then ϕ maps vertices to vertices and preserves the lengths of edges of $[AB]$.*

PROPOSITION 3.11. *Let $f : \mathbb{R}^3_T \to \mathbb{R}^3_T$ be an isometry such that $f(O) = O$. Then f is in O_h.*

PROOF. Let $A_1 = (1,0,0)$, $A_2 = (0,1,0)$, $A_5 = (0,0,1)$ and $D = (1,1,1)$. Consider $[OD]$ which is the rectangular prism (cube) with diagonal OD.

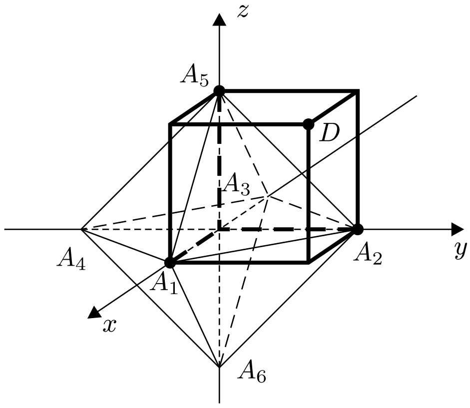

Fig. 4

It is clear from Fig. 4 that $f(A_1) \in A_iA_j$, $i \neq j$, $i, j \in \{1,2,3,4,5,6\}$. Here the points A_i and A_j are not on the same coordinate axis. Since f is an isometry by Corollary 3.10, $f(A_1)$, $f(A_2)$ and $f(A_5)$ must be the vertices of the rectangular prism with diagonal OD. Therefore, if $f(A_1) \in A_iA_j$, then $f(A_1) = A_i$ or $f(A_1) = A_j$. Similarly $f(A_2) = A_i$ or $f(A_2) = A_j$ and $f(A_5) = A_i$ or $f(A_5) = A_j$. Also any two of $f(A_1)$, $f(A_2)$ or $f(A_5)$ is not on the same coordinate axis. Now the following six cases are possible:

1. If $f(A_1) = A_1$, then $f(A_2) \in \{A_2, A_4\}$ and $f(A_5) \in \{A_5, A_6\}$ or $f(A_2) \in \{A_5, A_6\}$ and $f(A_5) \in \{A_2, A_4\}$.
2. If $f(A_1) = A_3$, then $f(A_2) \in \{A_2, A_4\}$ and $f(A_5) \in \{A_5, A_6\}$ or $f(A_2) \in \{A_5, A_6\}$ and $f(A_5) \in \{A_2, A_4\}$.
3. If $f(A_1) = A_2$, then $f(A_2) \in \{A_1, A_3\}$ and $f(A_5) \in \{A_5, A_6\}$ or $f(A_2) \in \{A_5, A_6\}$ and $f(A_5) \in \{A_1, A_3\}$.
4. If $f(A_1) = A_4$, then $f(A_2) \in \{A_1, A_3\}$ and $f(A_5) \in \{A_5, A_6\}$ or $f(A_2) \in \{A_5, A_6\}$ and $f(A_5) \in \{A_1, A_3\}$.
5. If $f(A_1) = A_5$, then $f(A_2) \in \{A_1, A_3\}$ and $f(A_5) \in \{A_2, A_4\}$ or $f(A_2) \in \{A_2, A_4\}$ and $f(A_5) \in \{A_1, A_3\}$.
6. If $f(A_1) = A_6$, then $f(A_2) \in \{A_1, A_3\}$ and $f(A_5) \in \{A_2, A_4\}$ or $f(A_2) \in \{A_2, A_4\}$ and $f(A_5) \in \{A_1, A_3\}$.

In each case it is easy to show that f is unique and is in O_h. For instance in the first case:

If $f(A_2) = A_2$ and $f(A_5) = A_5$, then f is the identity.
If $f(A_2) = A_2$ and $f(A_5) = A_6$, then $f = \sigma_\Delta$ such that $\Delta :\ z = 0$.
If $f(A_2) = A_4$ and $f(A_5) = A_5$, then $f = \sigma_\Delta$ such that $\Delta :\ y = 0$.
If $f(A_2) = A_4$ and $f(A_5) = A_6$, then $f = r_\pi$ with rotation axis $\parallel (1,0,0)$.
If $f(A_2) = A_5$ and $f(A_5) = A_2$, then $f = \sigma_\Delta$ such that $\Delta :\ y - z = 0$.
If $f(A_2) = A_5$ and $f(A_5) = A_4$, then $f = r_{\pi/2}$ with rotation axis $\parallel (1,0,0)$.
If $f(A_2) = A_6$ and $f(A_5) = A_2$, then $f = r_{3\pi/2}$ with rotation axis $\parallel (1,0,0)$.
If $f(A_2) = A_6$ and $f(A_5) = A_4$, then $f = \sigma_\Delta$ such that $\Delta :\ y + z = 0$.

The proofs of the remaining cases are quite similar to that of the first case. □

THEOREM 3.12. *Let $f : \mathbb{R}^3_T \to \mathbb{R}^3_T$ be an isometry. Then there exists a unique $T_A \in T(3)$ and $g \in O_h$ such that $f = T_A \circ g$.*

PROOF. Let $f(O) = A$ where $A = (a_1, a_2, a_3)$. Define $g = T_{-A} \circ f$. We know that g is an isometry and $g(O) = O$. Thus, $g \in O_h$ and $f = T_A \circ g$ by Proposition 3.11. The proof of uniqueness is trivial. □

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