Pyramidal Sections in Taxicab Geometry

The eccentricity definition of conics is borrowed and recycled, and Euclid is repaid with interest.

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The formula for the Euclidean distance between two points in the plane leads, early in the study of analytic geometry, to the definitions of the classical conic sections—the circle, ellipse, and hyperbola—in terms of distances to fixed points. Additionally, the parabola can be defined in terms of distances to a fixed point and a fixed line. After modest additional study in the area, the student becomes aware that the ellipse and hyperbola have comparable "point-line" definitions. It is this point-line approach to the conic sections I wish to pursue here—not in the context of Euclidean distance but using, instead, the **taxicab metric** defined in the coordinate plane by

$$d_T(P(x_1, y_1), Q(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

The d_T-distance between points P and Q is the length of a shortest path from P to Q composed of line segments parallel to the coordinate axes.

Previously published studies of the taxicab analogues of the conic sections [**1**], [**4**], [**5**], [**6**], [**7**] consider the plane curves whose definitions use the word "distance," interpret that word to mean the taxicab distance, and determine the shapes thus produced. In examining the ellipse and hyperbola from the standpoint of their "two-fixed-points" definitions, these studies obtain hexagons and octagons for their taxicab ellipses and an intriguing variety of unbounded graphs, some containing entire regions of the plane, for taxicab hyperbolas.

If the "point-line" definition is used instead, then some interesting results can be proved which show a strong analogy between the taxicab figures and the Euclidean conics. First, the taxicab figures obtained by using the focus-directrix definition of ellipse, parabola, and hyperbola can be obtained by projecting plane sections of a square pyramid onto a plane perpendicular to the axis of the pyramid. (The square pyramid is the natural object to section, since it is the taxicab analogue of the right circular cone.) Second, there is a simple way to locate the focus and directrix of these projections in the three-dimensional context of the sectioned pyramid. Third, this way of locating focus and directrix also works for the comparable projections of conic sections in Euclidean space.

Throughout our discussion, unless otherwise noted, points and lines are all contained in a fixed coordinate plane **P**. To begin our investigation, we need a taxicab definition of the distance from a point to a line. Let L be a line in the plane **P** and P a point in **P** not on L. The **taxicab distance** from P to L, denoted $d_T(P, L)$, is defined as

$$d_T(P, L) = \min\{d_T(P, Q) : Q \in L\}.$$

A couple of quick sketches show that $d_T(P, L)$ is very easy to compute or measure since, if L is a vertical line or if its slope m satisfies $|m| \geq 1$, then $d_T(P, L)$ is the length of the horizontal segment joining P to L; however, if $|m| \leq 1$, then $d_T(P, L)$ is the length of the vertical segment joining P to L. (It should be noted that such concepts as line, slope, and equation of a curve are coordinate concepts, dependent only on the coordinates of points, and are unaffected by our change to the taxicab metric.)

We now define the taxicab parabola, ellipse, and hyperbola, proceeding according to the

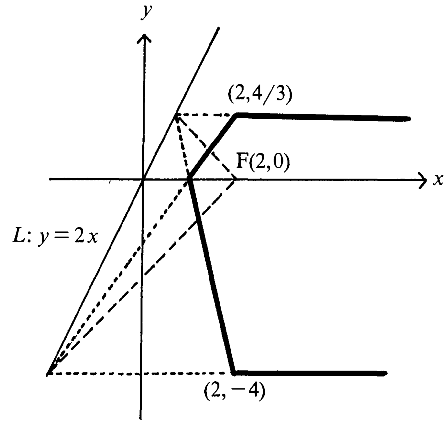

FIGURE 1. **Taxicab parabola (heavy line), general case, with directrix L: $y = 2x$ and focus $F(2,0)$; its equation is (1).**

analogous Euclidean prescription. Let F (the **focus**) be a fixed point in **P** and L (the **directrix**) be a fixed line in **P** which does not contain F. Then the set of all points P in the plane **P** such that $d_T(P, F)/d_T(P, L) = e$ (e is called the **eccentricity**) is called a **projected pyramidal section** (or **p-section**). The shape of the curve in **P** is determined by the ratio $d_T(P, F)/d_T(P, L) = e$, and we shall call the resulting plane figure an **ellipse** if $e < 1$, a **parabola** if $e = 1$, or a **hyperbola** if $e > 1$. We will justify the name "projected pyramidal section" by showing that these curves can also be obtained by sectioning a square pyramid with a plane and projecting that section onto another plane. Thus our choice of names for the three classes of curves is consistent with Euclidean geometry since the projection of a Euclidean ellipse, parabola, or hyperbola onto another suitably chosen plane is an ellipse, parabola, or hyperbola, respectively.

There is an inherent "asymmetry" in measurements using the taxicab metric; we have pointed out that the method of measuring distance from a point to a line depends on the slope of the line. For this reason, it is natural to consider three cases for the graphs of each p-section. If the directrix is a horizontal or vertical line in **P**, we shall refer to the graph as the **parallel case**; if the slope of the directrix is ± 1, we shall call the graph the **diagonal case**; the remaining possibilities will be the **general case**. We shall graph only one or two cases of each p-section here and thus leave the interested reader with a few cases to investigate.

We begin with the parabola. FIGURE 1 shows a graph of the general case with directrix L the line $y = 2x$ and focus the point $F(2,0)$. The defining relation $d_T(P, F)/d_T(P, L) = 1$ yields the equation of the parabola:

$$|x-2| + |y| = |x - y/2|. \tag{1}$$

The parabola can be sketched by dividing the plane into regions using the directrix and the horizontal and vertical lines through the focus as the separating lines, and graphing in each region the linear equation derived from (1) which is appropriate. (For example, in FIGURE 1, $|x-2| = 2-x$ for regions to the left of $F(2,0)$ and $|x-y/2| = x-y/2$ for regions to the right of the directrix.) Frequently, however, such a graph can be drawn more quickly using the knowledge that it consists of portions of straight lines and changes in direction only when it crosses the horizontal and vertical lines through the focus.

A parallel case of the parabola with directrix $x = -2$ and focus $F(2,0)$ is graphed in FIGURE 2, and a diagonal case of the parabola with directrix $y = -x - 4$ and focus $F(4,4)$ is graphed in FIGURE 3. The symmetry of the graphs in FIGURES 2 and 3 is typical, that is, the parallel and diagonal case will always have graphs symmetric to a line perpendicular to the directrix. Other graphs of taxicab parabolas appear in [**4**, p. 702], [**5**, p. 84], and [**7**, p. 147]; lattice-point graphs of the parallel and diagonal case appear in [**1**, p. 26].

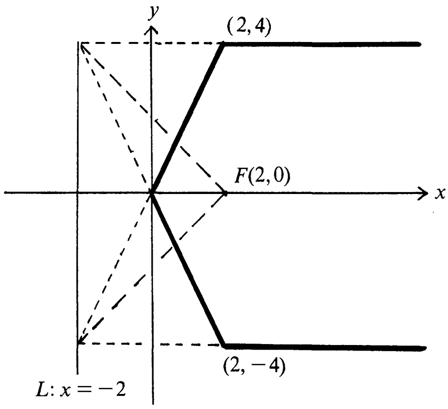

FIGURE 2. **Taxicab parabola (heavy line), parallel case, with directrix L: $x = -2$ and focus $F(2,0)$.**

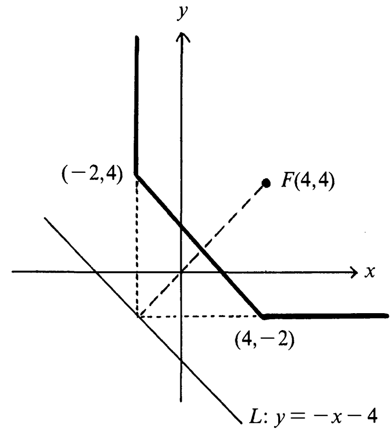

FIGURE 3. **Taxicab parabola (heavy line), diagonal case, with focus $F(4,4)$ and directrix L: $y = -x - 4$.**

Each taxicab ellipse is a quadrilateral. We show an example of a parallel case having the y-axis as directrix L, $F(3,0)$ as focus, and eccentricity $e = 1/2$. The general relation $d_T(P, F)/d_T(P, L) = 1/2$ gives the equation

$$|x-3| + |y| = \tfrac{1}{2}|x|, \tag{2}$$

which is graphed in FIGURE 4. In the parallel case, it can be proved that the Euclidean ellipse with the same directrix, focus, and eccentricity passes through all four vertices of the taxicab ellipse. Note that the taxicab ellipse in FIGURE 4 has one axis of symmetry, containing the focus and two vertices while the corresponding Euclidean ellipse has two axes of symmetry. The reader is encouraged to explore examples of the diagonal case and general cases, to find the similarities and differences of these taxicab ellipses.

The hyperbola assumes the most varied shapes of the p-sections of taxicab geometry. In the parallel and diagonal cases (as with these cases for the ellipse and parabola), the graph has one axis of symmetry. We leave these cases to be investigated by the reader. The general case is more complex, with the shape of the graph dependent on both the slope m of the directrix and the eccentricity e. There are three subcases. We illustrate the variation in shape of a taxicab hyperbola with focus $F(4,0)$ and eccentricity $e = 3$ for three different choices of directrix L with slope m: $|m| < e$, $|m| = e$, $|m| > e$. For the first subcase, we choose L as the line with equation $y = 2x$; the

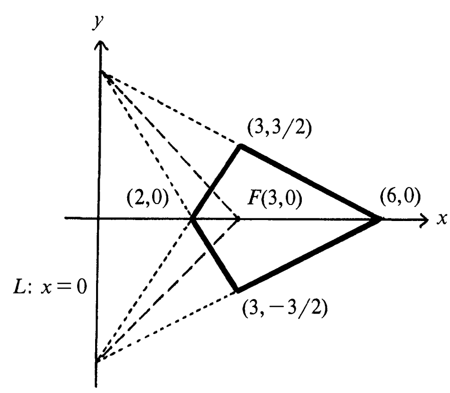

FIGURE 4. **Taxicab ellipse (heavy line), parallel case, with directrix L: $x = 0$ and focus $F(3,0)$; its equation is (2).**

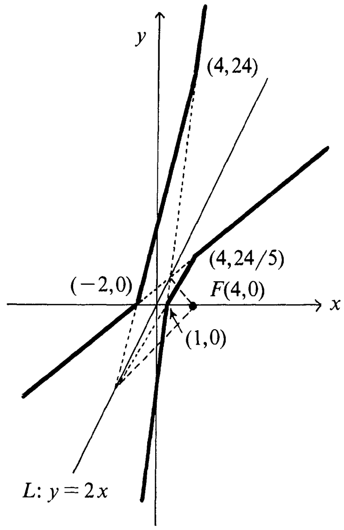

FIGURE 5(a). **Taxicab hyperbola (heavy line), general case with $|m|<e$. Here $e=3$, the focus is $F(4,0)$ and the directrix is L: $y=2x$. The equation of the graph is given by (3).**

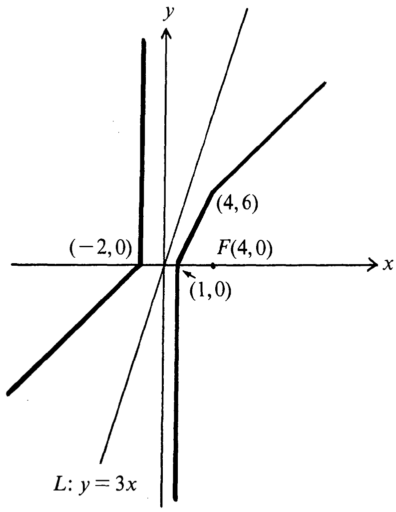

FIGURE 5(b). **Taxicab hyperbola with $e=3$, focus $F(4,0)$ and directrix L: $y=3x$ (case $|m|=e$).**

equation of this hyperbola is

$$|x-4|+|y|=3|x-y/2|, \tag{3}$$

and its graph is shown in FIGURE 5(a). If we increase $|m|$ while holding e constant, we find the lower ray of the right-hand branch of this hyperbola increasing in slope, and becoming vertical when $|m|=e=3$. At that instant the segment and upper ray of the left-hand branch also form a vertical ray. The graph of this hyperbola, whose equation is $|x-4|+|y|=3|x-y/3|$ is shown in FIGURE 5(b). If $|m|$ is increased so that $|m|>e$, then the upper ray on the left-hand branch assumes a negative slope as does the lower portion (no longer a ray) of the right-hand branch. This portion intersects the vertical line $x=4$ and bends, increasing in slope, as it crosses that line. A graph of this subcase, with $m=6$, is shown in FIGURE 5(c).

After examining the equations and graphs of several of the p-sections, it is easy to conjecture (and prove) the following general statement: *each p-section is the union of segments or rays from exactly four distinct lines (three for the diagonal parabola).* Far less obvious is the observation in Theorem 1 below. No doubt the curious reader has wondered why dashed and dotted lines occur in the graphs of FIGURES 1–5; this was done to illustrate the following fact.

THEOREM 1. *Two alternate sides of a p-section, if extended, will intersect at a point Q on the directrix of the section. A line of slope $+1$ or -1 through the focus of the section also passes through Q.*

The analytic proof of this theorem is composed of three direct computations, one for each case: parallel, diagonal, and general. Since geometers have been known to become ecstatic over the discovery of three-line incidence properties, this theorem, which asserts the incidence of four lines at the point Q, is superficially spectacular. However, this property turns out to be a routine consequence of the geometry of the sectioning of the square pyramid—to which we now turn our attention.

Consider a square pyramid (of two nappes) in three-dimensional space, the surface of which is the graph of the equation

$$|x|+|y|=c|z|, \quad c>0, \tag{4}$$

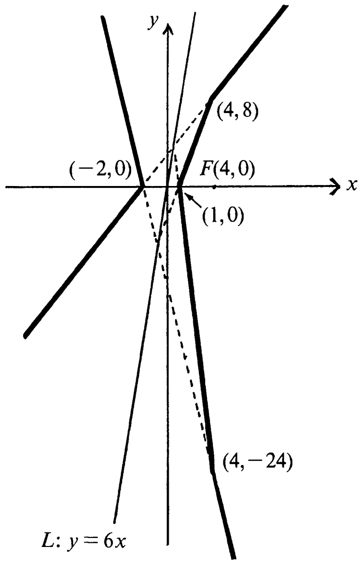

FIGURE 5(c). **Taxicab hyperbola with $e=3$, focus $F(4,0)$ and directrix L: $y=6x$ (case $|m|>e$).**

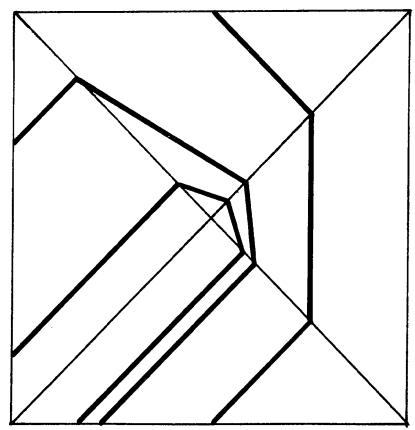

FIGURE 6. **The three cases of taxicab parabolas as projections of pyramidal sections.**

oriented for descriptive purposes with the z-axis vertical. It is not hard to prove that each of the cases (parallel, diagonal, general) of each of the basic shapes (parabola, ellipse, hyperbola) can be obtained from the intersection of a plane with the pyramid by perpendicular projection of that intersection onto the xy-plane. Even without detailed computation this seems plausible, since the equation of a plane is of the form

$$z = ax + by + d. \tag{5}$$

Solving (4) and (5) simultaneously by eliminating z (projecting onto the xy-plane) should produce an equation similar to (1), (2), or (3). It can be shown that, analogous to the prescription of Euclidean conic sections, a sectioning plane which is parallel to an edge of the pyramid will generate a parabola; a plane of lesser inclination, cutting all four edges of one nappe, will generate an ellipse; and one of greater inclination (thus cutting both nappes), a hyperbola.

FIGURE 5 illustrates the variations in shape of a hyperbolic p-section. By cutting the pyramid with a plane in such a way that the plane intersects both nappes *and the axis* of the pyramid, and projecting the section onto the xy-plane, you can achieve this variation in the components of the branches of the section by rotating the plane around the axis of the pyramid. The Euclidean hyperbola can, of course, be obtained by a section of the two nappes of a cone, but since Euclidean distances are unchanged by rotation, sections will not change shape if the sectioning plane is rotated around the axis of a right circular cone.

FIGURE 6 shows the three types of parabola as projections of pyramidal sections. (The reader is encouraged to slice a solid pyramid or draw the intersections of planes with the pyramid on its faces; the view from above the pyramid, looking down at the vertex, appears to the eye as the projections shown in FIGURE 6.) In the diagonal case, the sectioning plane is parallel to one face of the pyramid and in the parallel case the edge of the pyramid is equidistant from the two parallel edges of the section. In general, for all of the p-sections, the direction of the gradient of the sectioning plane σ (the vector in the xy-plane in the direction of the maximum inclination of σ) determines the case of each section. If the gradient is parallel to the x-axis or y-axis, then the pyramidal section will project onto the xy-plane as a parallel case; if the gradient has slope ± 1 in the xy-plane, the section projects as a diagonal case; other sections yield the general case.

We can illustrate these observations by showing how our earlier examples of p-sections are indeed projections of pyramidal sections. For this, we identify our coordinate plane **P** with the xy-plane. A parabolic p-section congruent (via the translation $x \to x+4$, $y \to y+4$) to that of FIGURE 3 is the projection of the intersection of the pyramid $|x|+|y|=2|z|$ with the plane σ whose equation is $x+y+6=2z$. (More generally, you can replace each "2" in these equations with an arbitrary $c>0$.) To verify this, consider the four equations of the planes $\pm x \pm y = 2z$ which form the faces of the pyramid, and eliminate the z-term between each and the equation of σ. The resulting equations in x and y which are consistent (one of them is not!) will be the equations of the projections of the lines of intersection of the pairs of planes into the xy-plane. Graph the portion of each line in its appropriate quadrant. Similarly, the p-section in FIGURE 1 is congruent (via the translation $x \to x-2$) to the projection onto the xy-plane of the intersection of the pyramid $|x|+|y|=|z|$ with the plane whose equation is $-2x+y+2z=4$. The p-section in FIGURE 4 is congruent (via the translation $x \to x-3$) to the projection of the intersection of the pyramid $|x|+|y|=\frac{1}{2}|z|$ with the plane whose equation is $x+z=-3$.

It should be noted that the p-sections usually are not themselves congruent to sections of a square pyramid. For example, the angle between the segment and a ray of the p-section in FIGURE 3 is 135°, but the angle between the corresponding parts of the pyramidal section whose projection is this p-section, is less than that. Also, in order for the graph in FIGURE 4 to be a pyramidal section would require the existence of a nondegenerate Euclidean right triangle with one leg equal to its hypotenuse.

The focus-directrix definition of a p-section in terms of the taxicab metric is an analytic one, and relies on the use of coordinates in the plane. However, the description of these curves as projections of sections of a square pyramid onto a plane π through the vertex of the pyramid and perpendicular to the axis of the pyramid is coordinate-free. But it is possible to locate the focus and directrix of the p-section in π. The illustration of parabolic p-sections shown in FIGURE 6 suggests an obvious location of the focus of these sections: the vertex of the pyramid.

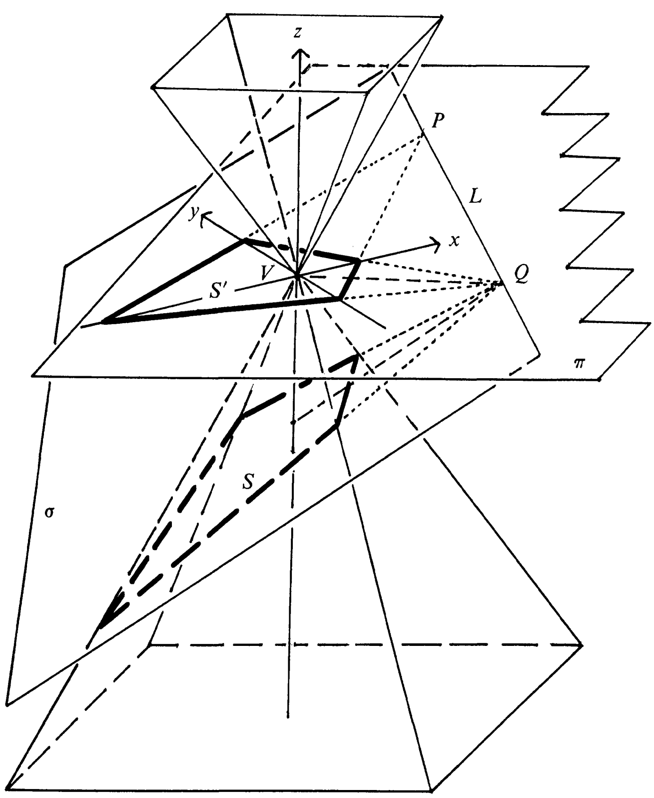

FIGURE 7. **Pyramidal section S and p-section S'.**

THEOREM 2. *Let π be a plane through the vertex V of a square pyramid, perpendicular to the axis of the pyramid. Let σ be a plane not perpendicular to π, which intersects π. Let S be the intersection of the pyramid with σ, and project S parallel to the axis of the pyramid onto the plane π; call this projection S'. Then S' is a p-section with focus V and directrix the line of intersection of the planes π and σ.*

FIGURE 7 illustrates Theorem 2 for the case of a p-section S' which is an ellipse. If we identify π with the xy-coordinate plane as shown in FIGURE 7 (the x and y axes are the projections of the edges of the pyramid onto π), then the fact that V is the focus of S' is immediately apparent, since V is the intersection of the x and y axes, which are the horizontal and vertical lines at which the graph of the p-section changes direction. (Note our earlier remarks about how to graph equation (1) in FIGURE 1.)

Theorem 1 says that the points of intersection of the lines containing alternate sides of a p-section are on its directrix; hence (except in the diagonal parabola case) the two points P and Q, thus determined by the two pairs of alternate sides, determine the directrix. But the lines through the sides of the p-section are just the projections into the xy-plane of the lines in which the sectioning plane σ intersects the face planes of the pyramid. FIGURE 8 shows that lines through alternate edges of the section S of the pyramid intersect in the plane π, so that this point of intersection coincides with the intersection of the projections of these lines in π. (In FIGURE 8,

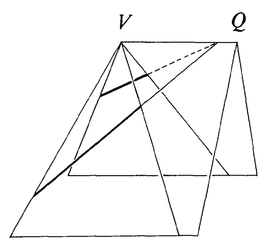

FIGURE 8

opposite faces of the pyramid have been extended to form a prism; the plane σ which cuts the pyramid in S has as its intersection with the prism lines through the edges of S, and clearly these lines intersect in the top edge of the prism, which lies in π.) Since P and Q are both in $\pi \cap \sigma$, and lie on the directrix of S', the line $\pi \cap \sigma$ is the directrix of S'. In the case of a diagonal parabola S', only one point Q on the directrix is determined as the intersection of alternate sides of S', so a different argument is needed to complete the proof that $\pi \cap \sigma$ is the directrix of S'. We leave this to the reader.

Since a search of available literature turned up no result comparable to Theorem 2 for the classic conics, it was natural to consider a comparable, and virtually identical, Euclidean theory of focus and directrix for projections of conic sections. Let π be a plane through the vertex V of a right circular cone K, with π perpendicular to the axis of K and consider any plane σ which is not parallel to π or to the axis of the cone. The intersection of σ and the cone K is a conic section S: an ellipse, parabola, or hyperbola. The projection of S parallel to the axis of the cone onto the plane π is a "fatter" ellipse, parabola, or hyperbola, respectively, which we shall call S'. The following theorem tells us where to find a focus and directrix of S'.

THEOREM 3. *In the above construction, the point V, the vertex of the cone K, is a focus of S', and the line of intersection of π and the sectioning plane σ is the directrix corresponding to this focus V.*

FIGURE 9 illustrates Theorem 3 for the case of an ellipse. The theorem can be proved analytically by direct calculation. Let the cone K have equation $c^2z^2 = x^2 + y^2$ and let the plane σ be given by $z = mx + b$. (The plane π is the xy-plane.) Eliminating z between these equations yields the equation of the projection S' of $S = K \cap \sigma$ into the xy-plane,

$$(1 - c^2m^2)x^2 - 2bc^2mx + y^2 = b^2c^2,$$

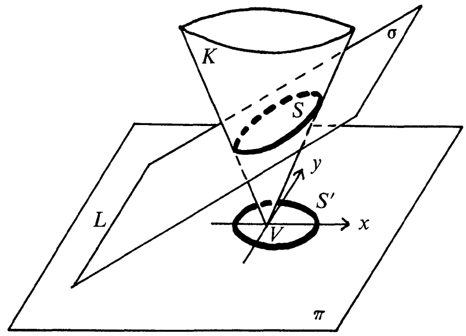

FIGURE 9. **Conic section *S* and projection *S′* with focus *V* and directrix *L*.**

which can be transformed into the standard equation of the conic. For example, in the case of the ellipse as in FIGURE 9 where $m < 1/c$, we obtain the standard form

$$(x-h)^2/A^2 + (y-k)^2/B^2 = 1$$

with

$$h = bc^2m/(1-c^2m^2), \qquad k=0, \quad A^2 = b^2c^2/(1-c^2m^2)^2,$$

and

$$B^2 = b^2c^2/(1-c^2m^2).$$

Thus the focal distance is C such that

$$C^2 = A^2 - B^2 = b^2c^4m^2/(1-c^2m^2)^2.$$

Since $h = C$, the left-hand focus is (0,0), the origin of the xy-plane (which is the vertex V of the cone K). The equation of the left-hand directrix L is $x = h - A^2/C$, which reduces to $x = -b/m$, the equation of the trace of the plane $z = mx + b$ in the xy-plane. The calculations for the hyperbola ($m > 1/c$) are comparable and those for the parabola ($m = 1/c$) are shorter. (In the cases of the ellipse and hyperbola we have an eccentricity $e = C/A = mc = m/(1/c)$, and we can conclude, additionally, that the eccentricity of S' will always be the ratio between the slope m of the sectioning plane and the slope $1/c$ of an element of the cone.)

There is also an interesting three-dimensional theory of focus and directrix of (unprojected) Euclidean conics, originally described by G. P. Dandelin in 1822 **[2]**, [3], **[8]**, which has no analogue in taxicab geometry. It gives both foci and both directrices for the ellipse and hyperbola using spheres inscribed in the cone; the theory presented in Theorem 3 gives only one focus and its directrix. In the case of the p-sections of taxicab geometry, however, there is only one focus and directrix because the figures lack the symmetry for two. It is unusual to be led to a result in Euclidean geometry by its analogue in a non-Euclidean geometry—particularly when the Euclidean theorem is simpler than the non-Euclidean.

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