Taxicab Geometry—A New Slant

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As anyone who has played with a bit of screen wire knows, a square is not a rigid geometrical figure. A slight push to the right or left will deform a square into a nonsquare rhombus. If such a transformation is performed on the points of a plane which has been coordinatized by a square grid so that the positive y-axis makes an angle of 60° with the positive x-axis and the shorter diagonals of each rhombus are drawn, an isometric grid results. Points will still be named by ordered pairs of real numbers with respect to the x-axis and the transformed y-axis (FIGURE 1a).

There are only three regular polygons which will tessellate the plane: the equilateral triangle, the square, and the regular hexagon. The first two of these can be subdivided into smaller similar polygons (yielding the isometric grid and the square grid). Work has already been done on taxicab geometry using the square grid; this note considers taxicab geometry using the isometric grid.

Square-taxi geometry arises because, for the two points $A=(x_1, y_1)$ and $B=(x_2, y_2)$, a new distance function is chosen:

$$d_S(A, B)=|x_1-x_2|+|y_1-y_2|.$$

Thus square-taxi geometry is not Euclidean because there the distance function, derived from the Pythagorean Theorem, is

$$d_E(A, B)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

In iso-taxi geometry three distance functions arise depending upon the relative positions of the points A and B. At the origin three axes occur: the x-axis, the y-axis and the y'-axis. This latter axis, one of the lines formed by the shorter diagonals of the rhombi mentioned earlier, forms an angle of 60° with the y-axis and with the x-axis. The three axes separate the plane into six regions called hextants. These hextants will be numbered I–VI in a counterclockwise direction beginning with the hextant where the coordinates of the points are both positive (FIGURE 1a). At any point in the plane three lines may be drawn parallel to the axes which, in turn, separate the plane into six regions. Two points could then have a I–IV orientation, a II–V orientation or a III–VI orientation to one another.

The three distance functions for iso-taxi geometry are

i) if the two points have a I–IV orientation (FIGURE 1c) then $d_I(A, B)=|x_1-x_2|+|y_1-y_2|$;
ii) if the two points have a II–V orientation (FIGURE 1b) then $d_I(A, B)=|y_1-y_2|$;
iii) if the two points have a III–VI orientation (FIGURE 1b) then $d_I(A, B)=|x_1-x_2|$.

If the two points lie on a line parallel to the x-axis, then formula iii) is used; if the two points lie on a line parallel to the y-axis or to the y'-axis then formula ii) is used.

The reader may observe that the length of the segment $\overline{AB}$ can be quickly found by counting the number of units along the two adjacent sides of the parallelogram ($\square\, AQBP$ in FIGURE 1c), having $\overline{AB}$ as its longer diagonal, formed by lines parallel to the axes drawn at each endpoint of the segment.

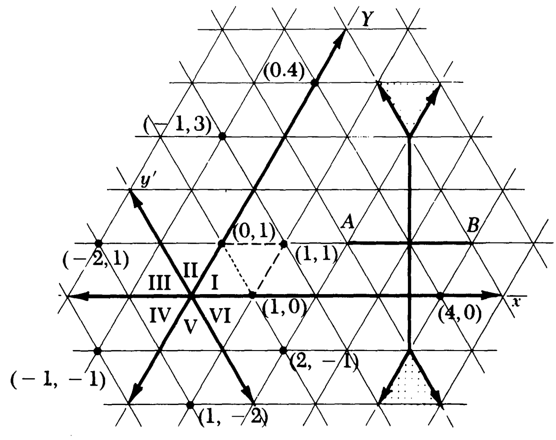


FIGURE 1a

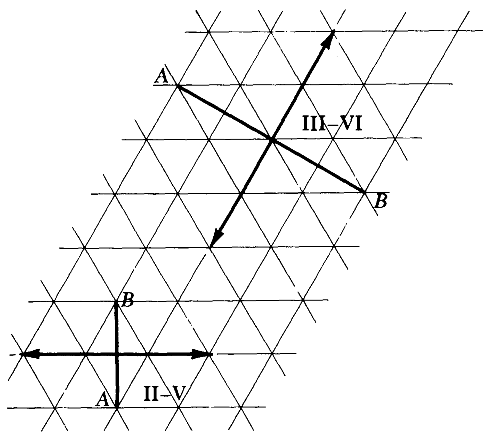


FIGURE 1b

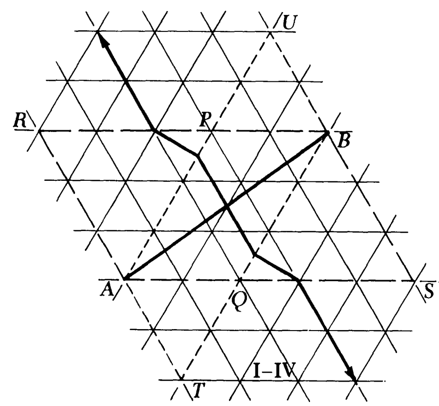


FIGURE 1c

As in Euclidean geometry and in square-taxi geometry, it can be shown that for any two points A and B

a) $d_I(A, B) \geqslant 0$;
 $d_I(A, B) = 0$ if and only if $A = B$;
b) $d_I(A, B) = d_I(B, A)$;
c) if X is any point in the plane,
 $d_I(A, X) + d_I(X, B) \geqslant d_I(A, B)$;
 if equality holds and A, X and B are distinct and collinear, then X is "between" A and B, and conversely.

In iso-taxi geometry two segments are defined to be congruent if and only if they have the same iso-taxi length. The transformations of the plane which preserve iso-taxi distances are reflections about any line such that the angle it makes with the horizontal is a multiple of 30°, all translations; and rotations about any point such that the angle of rotation is a multiple of 60°.

Because angle measure is not dependent upon the distance function, angles may be measured as they are in both Euclidean and in square-taxi geometry. Only Euclidean plane geometry has a rotation invariant metric; therefore one should expect properties involving angle measurement to be altered in taxicab geometries. Specifically, the side-angle-side axiom may be assumed in neither iso-taxi geometry nor in square-taxi geometry. Of course, any theorem which relies on the side-angle-side axiom is also invalid in each of these taxicab geometries. Among these are the angle-side-angle theorem, the angle-angle-side theorem and the side-side-side theorem. The reader may wish to find counterexamples in iso-taxi geometry for the axiom and for each of the theorems. Also, an isosceles triangle may no longer have its base angles congruent and triangles exist where the sum of the lengths of two of the sides equals the length of the third side.

Special geometric sets The *iso-taxi circle* is a hexagon. Because the circumference of the iso-taxi circle is six times the radius, $pi_I = 3$. The square-taxi circle is a square and $pi_S = 4$. It should be observed that each of these values for *pi* is the extreme value obtainable when an affinely regular hexagon is used (iso-taxi) or when a parallelogram is used (square-taxi); other values must fall between these extremes [**6**].

Using an equilateral triangle (whose sides are parallel to the axes and are of one unit in length) as a unit of area, we can find areas of figures in iso-taxi geometry by tessellating the region to be measured with these "equi-tri" units and counting the number required. The formula for the area of an iso-taxi circle is $A = 2 * pi_I * r^2$ "equi-tri" units, whereas the formula for the area of a square-taxi circle is $A = .5 * pi_S * r^2$ square units.

The set $\{X | d_I(A, X) + d_I(X, B) = d_I(A, B)\}$ might be called the *iso-taxi minimum distance set*. It consists of all points in the interior and on the parallelogram ($\square$ $AQBP$ in FIGURE 1c) determined by lines parallel to the axes drawn at the points A and B which has $\overline{AB}$ as its longer diagonal. The minimum distance set in square-taxi geometry consists of all points in the interior and on the rectangle having sides parallel to the axes and having the segment as a diagonal. In each case, if the two points are on a line parallel to an axis then the minimum distance set is a segment.

The set of points in a plane equidistant from two fixed points may be called a mid-set of the segment determined by the two fixed points. Points of the mid-set may be found by locating points of intersection of circles of equal radii having A and B as centers.

The *iso-taxi mid-set* $\{X | d_I(A, X) = d_I(X, B)\}$, as well as the square-taxi mid-set, may assume different shapes depending on the relative positions of the endpoints of the segment (FIGURE 1). The three cases are

a) The points A and B lie on a line parallel to an axis.
   i) In iso-taxi geometry the mid-set is a perpendicular segment and two regions (FIGURE 1a).
   ii) In square-taxi geometry the mid-set is the perpendicular bisector of the segment.
b) The points A and B lie on a line which forms an angle with the x-axis whose measure is
   i) 30° or 90° (iso-taxi). The mid-set is the perpendicular bisector of the segment (FIGURE 1b).
   ii) 45° (square-taxi). The mid-set is a perpendicular segment and two regions.
c) The points A and B lie on a line different from those of a) or b).
   i) In iso-taxi geometry the mid-set consists of three segments and two rays (FIGURE 1c). The "breaks" occur on the edges of the two parallelograms, $ASBR$ and $ATBU$, determined by lines parallel to the axes drawn at the points A and

B, which have $\overline{AB}$ as their shorter diagonal.

ii) In square-taxi geometry the mid-set consists of one segment and two rays. The "breaks" occur on the edges of the minimum distance set.

An ellipse is the set of points in a plane such that the sum of its distances from two fixed points (foci) is a constant. Points of the ellipse may be found by locating points of intersection of circles having A and B as centers such that the sum of the lengths of their radii equals the constant.

The *iso-taxi ellipse*, $\{X | d_I(A, X) + d_I(X, B) = k\}$ where $k > d_I(A, B)$, also assumes different shapes for each of the three cases enumerated above and for different values of the constant sum, k. For example, by considering sets of confocal ellipses, one observes (FIGURE 2) that

in case a) the iso-taxi ellipse is a decagon except when k equals twice the distance between the foci, then it becomes an octagon;

in case b), it is a dodecagon except when k equals one and one-half the distance between the foci; then it is an octagon;

in case c), it is a dodecagon except in two cases in which it becomes a decagon.

It is interesting to observe that the exceptional cases occur at points of intersection of lines parallel to the axes, drawn at each foci. Also when these lines are drawn, the plane is separated into regions and, in each case, segments of the confocal ellipses are parallel in each, or all but one, of these regions. In cases a) and b) there exist two lines of symmetry; in case c) there exists only point symmetry.

In square-taxi geometry ellipses are octagons in cases b) and c) but they are hexagons in case a). Different values of the constant do not affect the shape of the curves. If lines parallel to the axes are drawn at each vertex, parallel segments of the ellipse occur in each region in case a) and in all but one of the regions in the latter two cases. In each case there exist two lines of symmetry—one vertical, one horizontal.

A hyperbola is the set of points in a plane such that the difference of its distances from two fixed points is a constant. Points of the hyperbola may be found by locating points of intersection of circles having A and B as centers such that the difference of the lengths of their radii equals the constant.

The *iso-taxi hyperbola*, $\{X | |d_I(A, X) - d_I(X, B)| = k\}$ where $0 < k < d_I(A, B)$, also assumes different shapes for each of the three cases and for different values of k

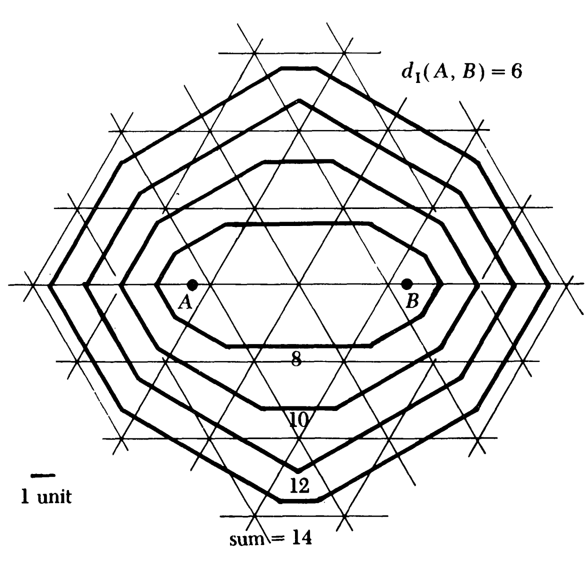


FIGURE 2a

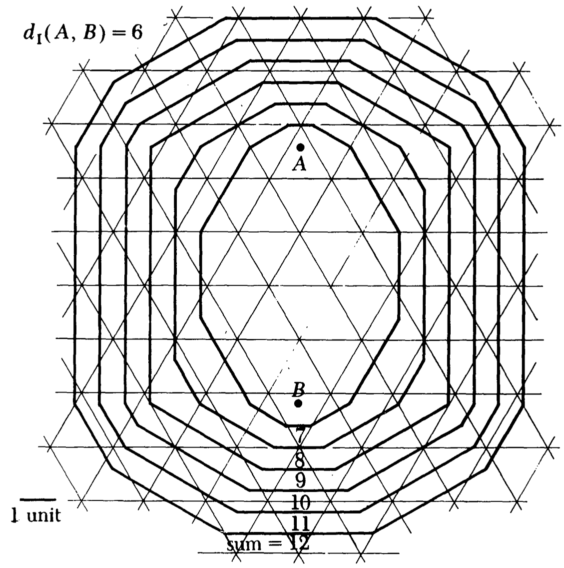


FIGURE 2b

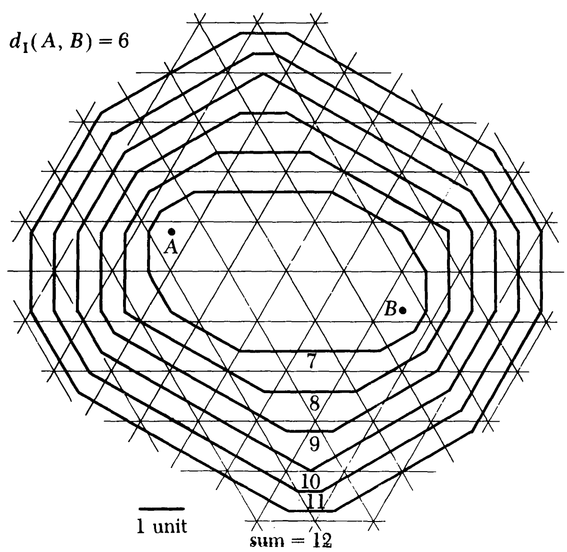


FIGURE 2c

in two of them. For example, by considering confocal hyperbolas, one observes (FIGURE 3) that

in case a) each wing of an iso-taxi hyperbola consists of a segment and two rays;

in case b), for $k < .5 * d_I(A, B)$ each wing consists of three segments and two collinear rays, for $k > .5 * d_I(A, B)$ each wing consists of three segments and two noncollinear rays, and for $k = .5 * d_I(A, B)$ each wing consists of three segments and two regions;

in case c) many different configurations occur—a wing of the iso-taxi hyperbola may consist of three segments and two parallel rays, or of a region, a segment and a ray, or of three segments and two nonparallel rays, or of a region, three segments and a ray.

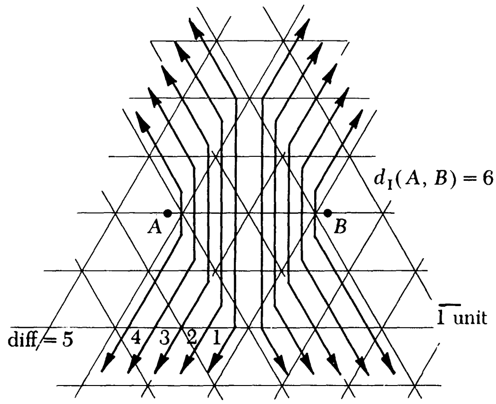


FIGURE 3a

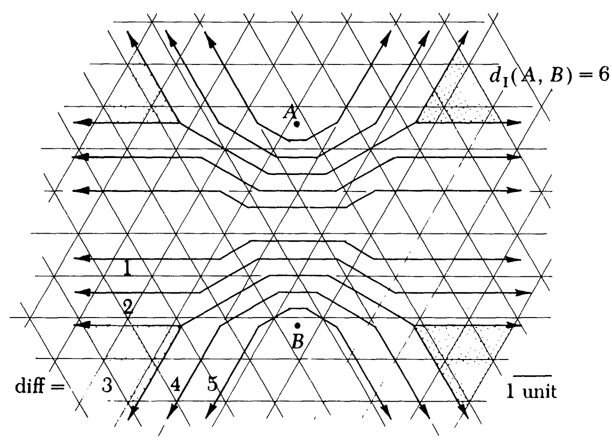


FIGURE 3b

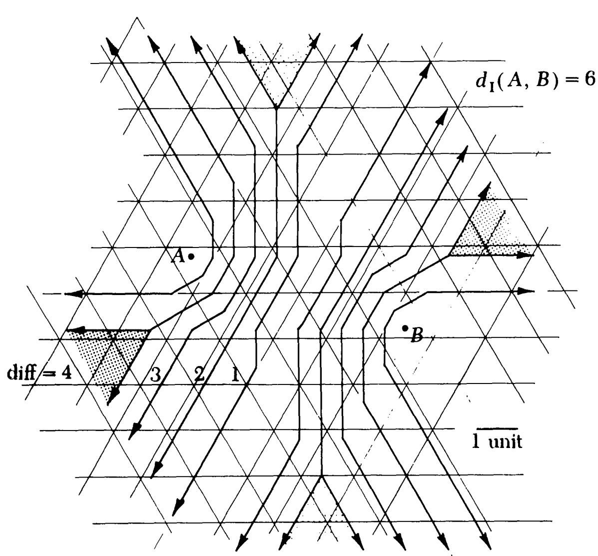


FIGURE 3c

Again, if at each focus lines parallel to the three axes are drawn then parallelism of segments or of rays is observed in all but six of the regions in case a) and in all but two of the regions in cases b) and c)—or the entire region becomes a part of the graph of an iso-taxi hyperbola. Two lines of symmetry occur in the first two cases; a point of symmetry occurs in the last.

For a square-taxi hyperbola one observes that

in case a) each wing is a straight line;

in case b) each wing consists of a segment and two rays;

in case c) each wing may consist of a segment and two parallel rays, or a region, a segment and a ray, or a segment and two nonparallel rays.

If lines parallel to the axes are drawn at each vertex, parallel segments occur, parallel rays occur, or a region of the hyperbola occurs in all but two of the regions in case c) and in all but four of the regions in the former two cases. Symmetry patterns for the three cases are the same as those of their iso-taxi counterparts.

The distance from a point to a line is the length of the radius of a circle, having the point as its center, which is tangent to the line. Thus, in iso-taxi geometry (FIGURE 4), or in square-taxi geometry, this distance is the length of the shortest segment to the line in a direction parallel to an axis.

The set of points in a plane equidistant from a line consists of two lines parallel to the given line. The use of parallel lines is important in locating points of parabolas and points of angle mid-sets.

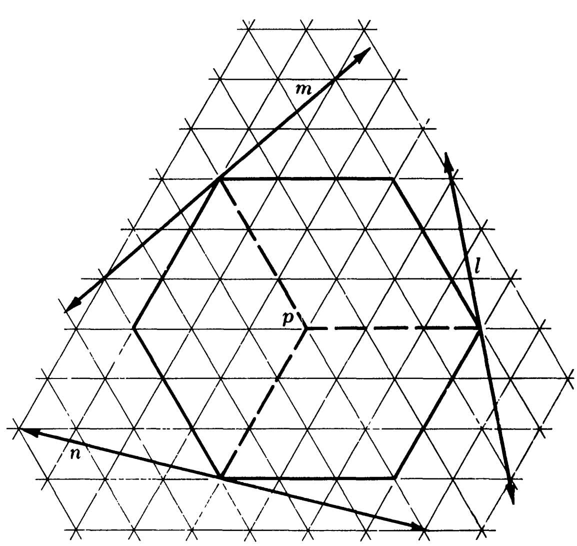


FIGURE 4

The set of points in a plane equidistant from a fixed point (focus) and a fixed line (directrix) is a parabola. Points of a parabola may be found by locating points of intersections of circles having their center at the focus and radius k, and lines parallel to the directrix and k units distant from it, $k \geqslant .5 * d$ where d is the distance from the focus to the directrix. The set where $k = .5 * d$ is called the vertex of the parabola.

The *iso-taxi parabola*, $\{X | d_I(F, X) = d_I(X, \overleftrightarrow{AB})\}$ where F is the focus and the directrix is determined by the points A and B, assumes different shapes depending on the relative positions of the points A and B. Families of parabolas having the same directrix and different foci are considered in each of the three cases (FIGURE 5). Each iso-taxi parabola in case a) consists of three segments and two nonparallel rays; the vertex, in this case, is one of these segments. In cases b) and c) the vertex is a point and the parabola consists of four segments and two parallel rays. In the first two cases there is a symmetry about the line containing the focus that is perpendicular to the directrix. In case c) all symmetry is lost. The "break" points of the parabola in each case may be observed to lie on lines parallel to the axes drawn at the focus.

In square-taxi geometry a parabola consists of a segment (the vertex) and two nonparallel rays in case b) whereas in the other two cases it consists of two segments and parallel rays, and the vertex is a point. Symmetry and "break" points occur as they do for iso-taxi parabolas.

The set of points in a plane equidistant from two noncollinear rays having a common endpoint may be called the angle mid-set. Points of the angle mid-set may be found by locating points of intersection of lines equidistant from the lines containing

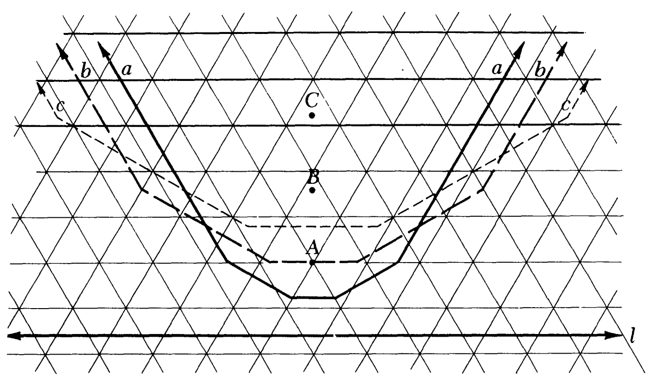
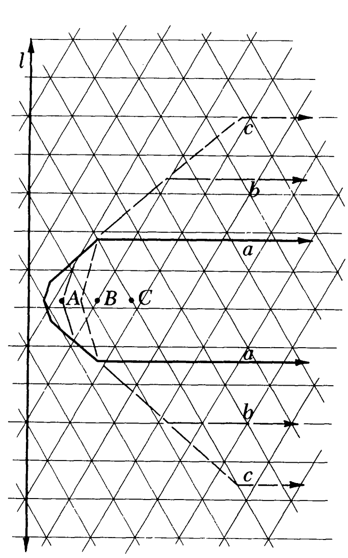


FIGURE 5a



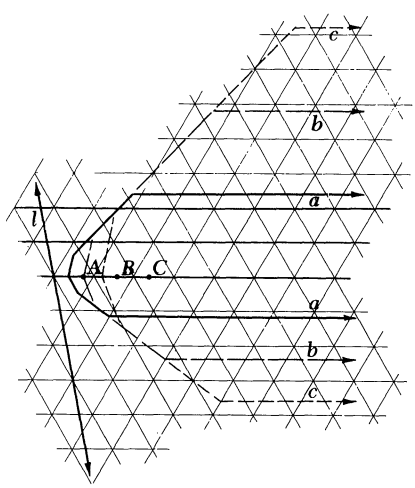


FIGURE 5b

FIGURE 5c

the rays of the angle; these points must also lie in the interior of the angle. The angle mid-set is, therefore, a half-line. Points of the angle mid-set are also the centers of circles which are tangent to each of the two rays.

The *iso-taxi angle mid-set*, $\{X|d_I(X,\overrightarrow{BA})=d_I(X,\overrightarrow{BC})\}$, is the same as the Euclidean angle bisector only for especially placed angles (for example, angles whose rays are symmetric to a line forming an angle with the x-axis such that its measure is a multiple of 30°). The iso-taxi angle mid-set is unique for any given angle; however, if the angle is especially placed then the rays may be tangent to the iso-taxi circles only at the vertex of the angle (for example, if the angle has a ray parallel to an axis and the measure of the angle is greater than or equal to 120°). Consequently, different angles having a common ray may have the same iso-taxi angle mid-set. In FIGURE 6 each of the angles $\angle ABC$, $\angle ABD$ and $\angle ABE$ has $\overrightarrow{BF}$ as its angle mid-set. Also iso-taxi angle mid-sets of two pairs of vertical angles may not be perpendicular.

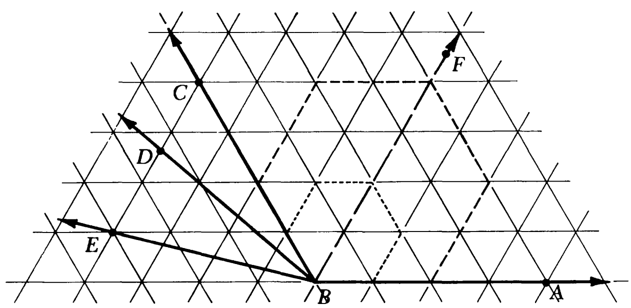


FIGURE 6

The same conclusions are true for square-taxi angle mid-sets except in the first case, the measure of the angle is a multiple of 45°, and in the second case, the angle has a ray which is contained in a line which forms an angle with the x-axis whose measure is 45° and the angle, itself, has measure greater than or equal to 90°.

Because of the uniqueness of the iso-taxi angle mid-set, the incenter of a triangle and its corresponding inscribed circle are also unique. However, because of the observations noted above, the inscribed circle may be tangent to a side of the triangle at a vertex of the triangle (FIGURE 7). Similar results arise in square-taxi geometry.

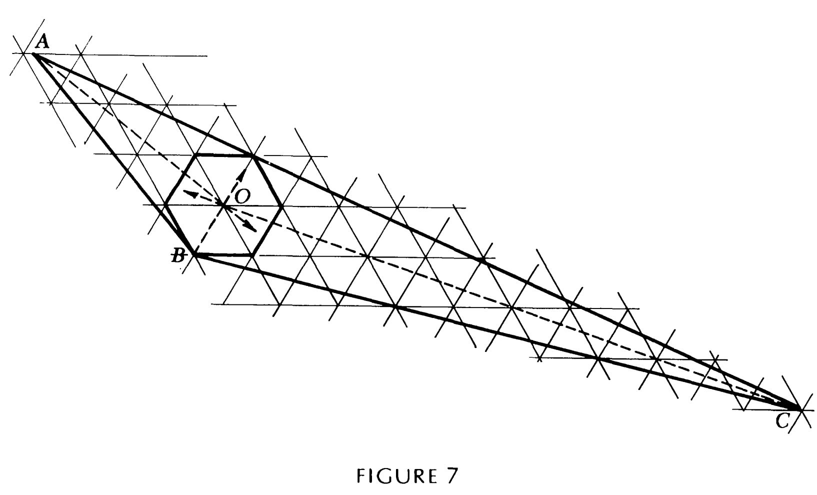


FIGURE 7

The circumcenter of a triangle and its corresponding circumcircle may not be unique in taxicab geometry. In fact the set of circumcenters may consist of a point, a segment, a segment and a ray, or a ray. In iso-taxi geometry a segment occurs if one side is parallel to an axis and the other sides are congruent and more than twice the length of the first side (FIGURE 8a); a segment and a ray occur if two sides are parallel to the axes and the length of the third side equals the sum of the lengths of the other two sides (FIGURE 8b); and a ray occurs if two congruent sides are parallel to the axes and the length of the third side equals the sum of the lengths of the other two sides (FIGURE 8c). In square-taxi geometry a segment occurs if exactly one side is a segment of a line making an angle with the x-axis whose measure is 45° and the other two sides are congruent and greater in length than the first side; a segment and a ray occur if two sides are segments of such 45° lines and the triangle is isosceles but not equilateral; and a ray occurs if at least one side is a segment of a 45° line and another side is parallel to an axis.

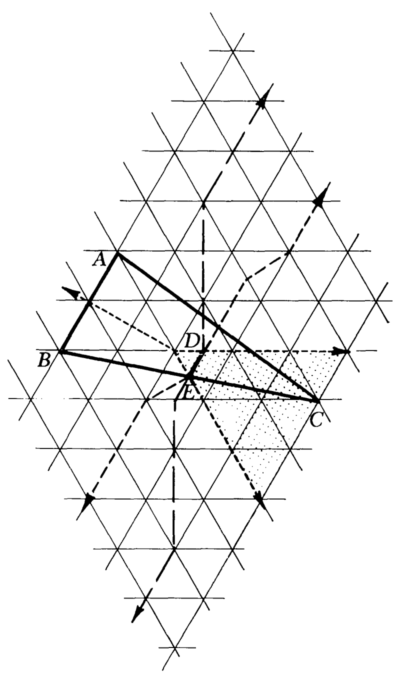


FIGURE 8a

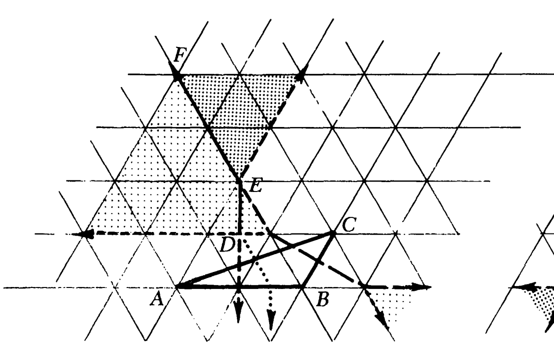


FIGURE 8b

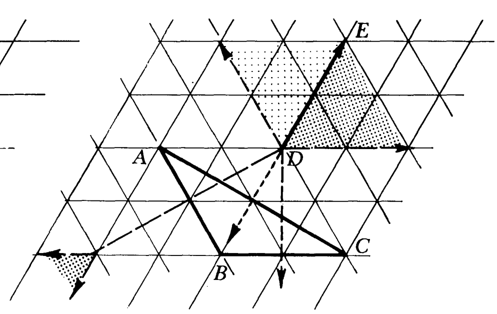


FIGURE 8c

Other taxicab geometric sets might also be explored and additional theorems established as valid or invalid for this geometry. Surely changing the distance function and changing the coordinate grid from a square configuration to a triangular configuration leads to geometries that behave very differently from Euclidean geometry.

The author expresses her appreciation to Mr. Ron Spangler, ECU, for his assistance in the preparation of the illustrations.

REFERENCES

1. Martin Gardner, Mathematical games, *Scientific American* 243 (Nov. 1980), 18–30.
2. Eugene F. Krause, Taxicab geometry, *Mathematics Teacher* 66 (1973), 695–706.
3. ——, *Taxicab Geometry*, Addison-Wesley, Menlo Park, NJ, 1975.
4. Richard Laatsch, Pyramidal sections in taxicab geometry, this MAGAZINE 55 (1982), 205–212.
5. Barbara E. Reynolds, Taxicab geometry, *Pi Mu Epsilon J.* 7 (1980), 77–88.
6. J. J. Schaffer, *Geometry of Spheres in Normed Spaces*, Marcel Dekker, Inc., New York, 1976, pp. 17–18.
7. Doris J. Schattschneider, The taxicab group, *Amer. Math. Monthly* 91 (1984), 423–428.

The Axis of a Rotation: Analysis, Algebra, Geometry

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Introduction Recently, while working on some problems related to coordinate transformations, I happened on the following discovery:

> If the 3 by 3 matrix A represents a rotation (i.e., A is orthogonal with determinant 1), and the trace of A is $\operatorname{tr}(A)$, then for any vector $\mathbf{x}$
>
> $$A\mathbf{x} + A^T\mathbf{x} + [1 - \operatorname{tr}(A)]\mathbf{x}$$
>
> lies on the axis of the rotation.

I recognized immediately that this result must be well known (in certain circles). However, it seems to me that my route of discovery illustrates some important principles of problem solving and mathematical discovery. I present this account with the idea in mind that a suitable modification might be presented in a linear algebra course. In addition to serving as a case study in discovery, the topic is a natural application of eigenvalues and eigenvectors, and the result has an attractive simplicity. As suggested by the title, analysis, algebra, and geometry each play a role in the development to follow.

Background Before proceeding further, it will be useful to review some facts about rotation matrices and to establish the notation and nomenclature to be used. A