Taxi Cab Geometry: History and Applications

Chip Reinhardt
Stevensville High School, Stevensville, Montana

Paper submitted: December 9, 2003 Accepted with revisions: 27. July 2004

We will explore three real life situations proposed in Eugene F. Krause's book *Taxicab Geometry*. First a dispatcher for Ideal City Police Department receives a report of an accident at X = (-1,4). There are two police cars located in the area. Car C is at (2,1) and car D is at (-1,-1). Which car should be sent? Second there are three high schools in Ideal city. Roosevelt at (2,1), Franklin at (-3,-3) and Jefferson at (-6,-1). Draw in school district boundaries so that each student in Ideal City attends the school closet to them. For the third problem a telephone company wants to set up payphone booths so that everyone living with in twelve blocks of the center of town is with in four blocks of a payphone. Money is tight, the phone company wants to put in the least amount of payphones possible such that this is true.

> What makes these problems interesting is that we want to solve them not as the "crow flies", but with the constraints that we have to stay on city streets. This means the distance formula that we are accustom to using in Euclidean geometry will not work. Fortunately there is a non Euclidean geometry set up for exactly this type of problem, called taxicab geometry. This system of geometry is modeled by taxicabs roaming a city whose streets form a lattice of unit square blocks (Gardner, p.160).

Hermann Minkowski first seriously proposed taxicab geometry around the turn of the century. He was born in Russia and was young Albert Einstein's teacher in Zurich (Gardner, p160). Hermann published a whole family of "metrics", that is, examples of spaces in which a way of measuring distance is defined so that it fulfills the axioms of a metric space. Among these metrics is one that is referred to as taxicab metric. The distance formula in this metric is the same used today in taxicab geometry. (Reynolds, 77). It was not until the 1952, when Karl Menger established a geometry exhibit at the Museum of Science and Industry of Chicago that taxicab geometry actually got its name. Accompanying the exhibit was a booklet, entitled *You Will Like Geometry*, in which the term "taxicab" geometry was first used (Golland, 326).

Taxicab geometry is a non-Euclidean geometry that is accessible in a concrete form and is only one axiom away from being Euclidean in its basic structure. The points are the same, the lines are the same, and angles are measured the same way. Only the distance function is different. In Euclidean geometry distance between two points P and Q are defined by:

$d(P,Q) = ((x_1,y_1),(x_2,y_2))$
$d(P,Q) = \sqrt{((x_2-x_1)^2 + (y_2-y_1)^2)}$

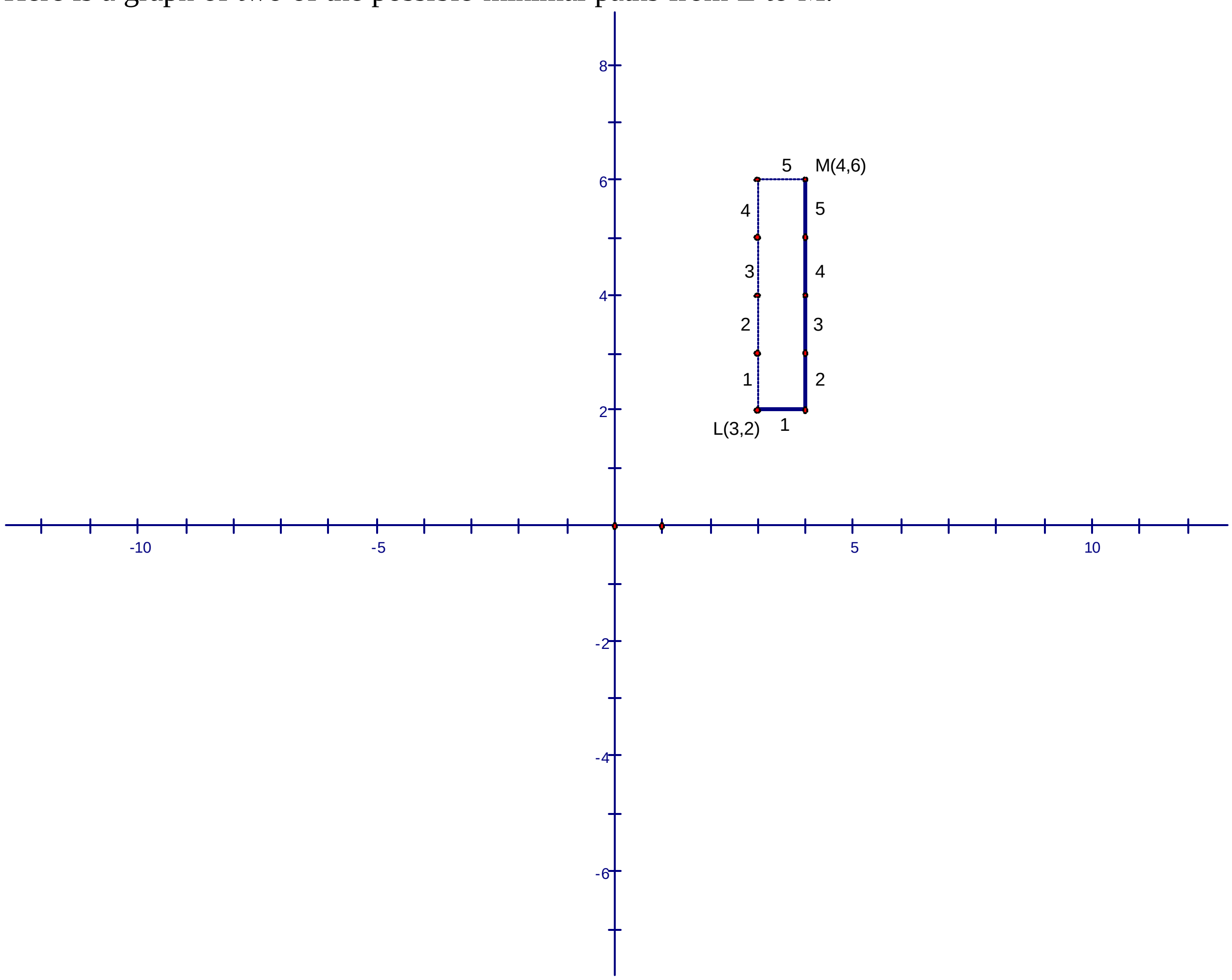
The minimum distance between two points is a straight line in Euclidean geometry.. In taxicab geometry there may be many paths, all equally minimal, that join two points. Taxicab distance between two points P and Q is the length of a shortest path from P to Q composed of line segments parallel and perpendicular to the x-axis. We use the formula:

$d(P,Q) = [(x_1,y_1),(x_2,y_2)]$
$d(P,Q) = |x_2-x_1| + |y_2-y_1|$

Lets look at the distance between points L(3,2) and M(4,6)

$d(L,M) = |4-3| + |6-2|$
$= |1| + |4|$
$= 5$

Here is a graph of two of the possible minimal paths from L to M.



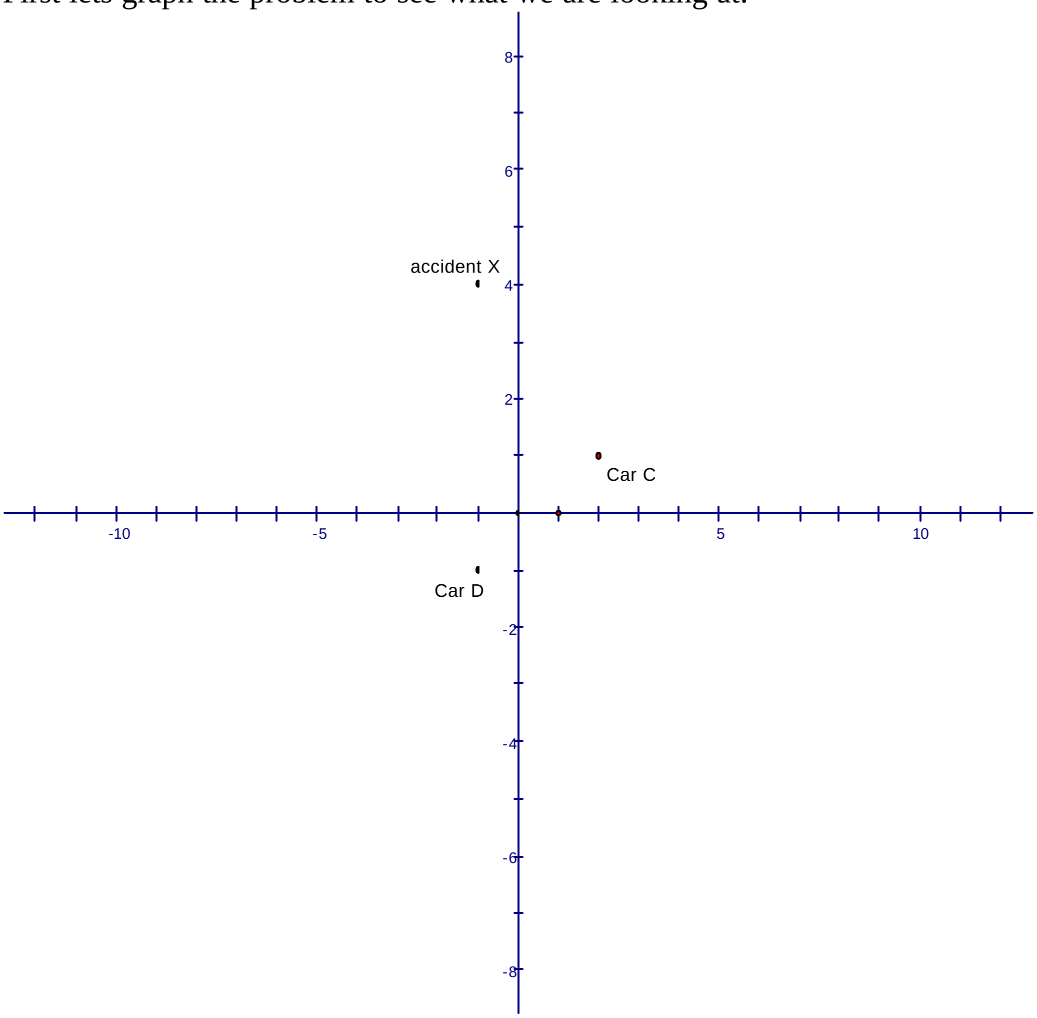
There are three more paths that are a distance of five. Can you find them?

Using our new distance formula, we will now set out to find the solutions to the problems proposed at the beginning of this paper.

Problem One

A dispatcher for Ideal City Police Department receives a report of an accident at X = (-1,4). There are two police cars located in the area. Car C is at (2,1) and car D is at (-1,-1). Which car should be sent?

First lets graph the problem to see what we are looking at:



The police cars cannot drive through peoples' houses. They have to stick to the streets. Taxicab geometry will be the best choice to solve this problem. One simply needs to compare the distance in taxicab geometry from the dispatcher to each patrol car.

The distance between accident X and car C is:

$$
\begin{aligned}
d(X,C) &= [(-1,4),(2,1)] \\
&= |2-(-1)| + |1-4| \\
&= |3| + |-3| \\
&= 3 + 3 \\
&= 6
\end{aligned}
$$

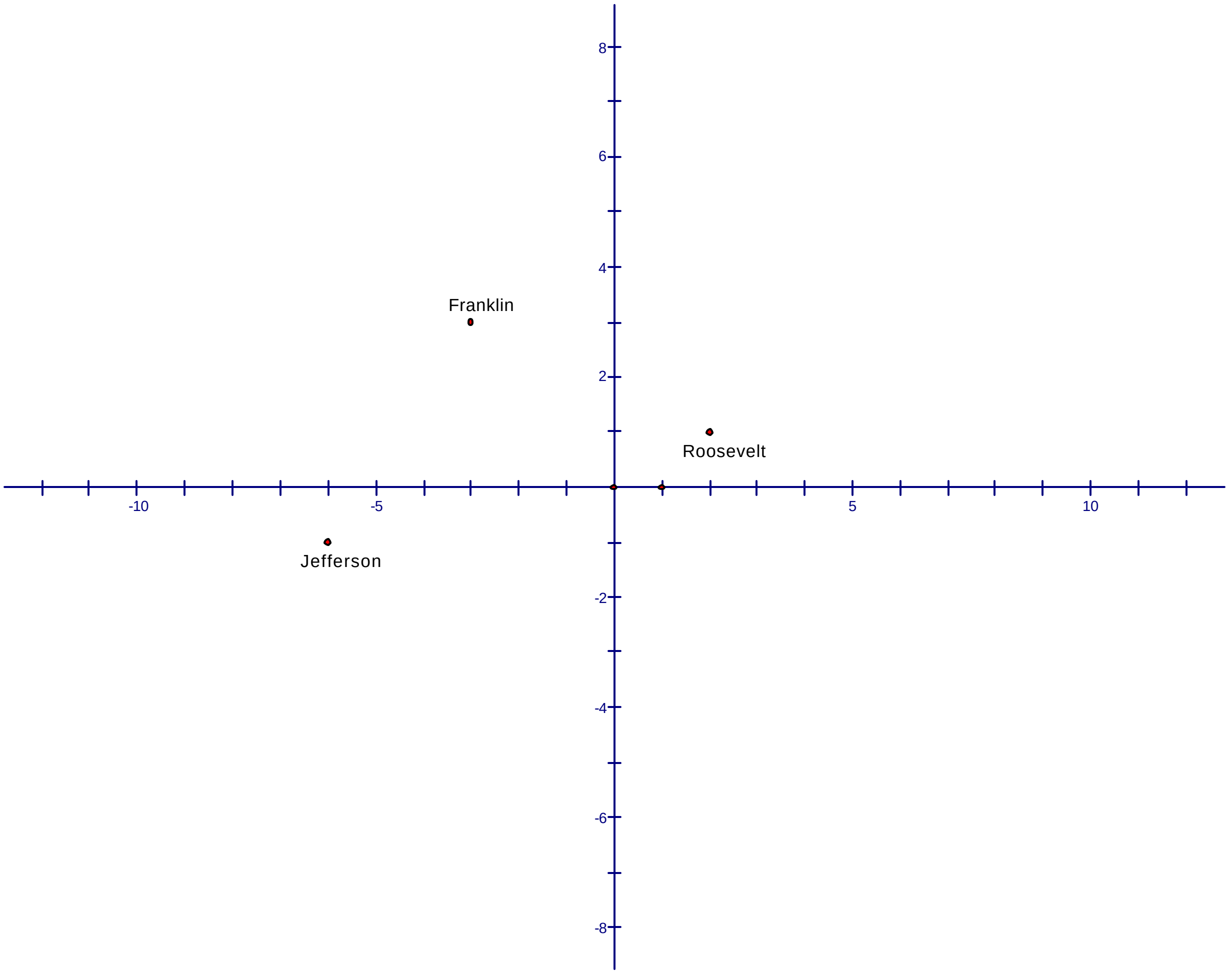
The distance between accident X and car D is:

$$
\begin{aligned}
d(X,D) &= [(-1,4),(-1,-1)] \\
&= |-1-(-1)| + |-1-4| \\
&= |0| + |-5| \\
&= 0+5 \\
&= 5
\end{aligned}
$$

Thus we can clearly see that car D is one block closer to accident X.

Problem Two

We want to draw school district boundaries such that every student is going to the closest school. There are three schools: Jefferson at (-6, -1), Franklin at (-3, -3), and Roosevelt at (2,1).



Taxicab geometry will be a logical choice to find the solution because students will have to stick to the streets when traveling to school. The solution to this problem can not be found at once. The problem needs to be broken down into sections.

Section I

Lets first focus on a boundary between Franklin and Jefferson school. The boundary line needs to fall on the points were the distance between Jefferson and Franklin are the same. So we need:

$$d(\text{Jefferson}) = d(\text{Franklin})$$
$$d[(-6,-1),(x,y)] = d[(-3,3),(x,y)]$$
$$|x+6| + |y+1| = |x+3| + |y-3|$$

Solving for x and y becomes more difficult with the absolute values. We need to look at the cases were $x+6<0 \Rightarrow x<-6$, $y+1<0 \Rightarrow y<-1$, $x+3<0 \Rightarrow x<-3$, and $y-3<0 \Rightarrow y<3$. This translates into nine different cases:

	$-1 \le y \le 3$	$y < -1$	$y > 3$
$-6 \le x \le -3$	case I	case IV	case VII
$x < -6$	case II	case V	case IIX
$x > -3$	case III	case VI	case IX

We look at these cases because the absolute values will change the solutions.

<u>Case I</u>: $-1 \le y \le 3$ and $-6 \le x \le -3$

Since $|x+6| \ge 0$ when $-6 \le x \le -3$, $|x+6| = x+6$
$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$
$|x+3| \le 0$ when $-6 \le x \le -3$, $|x+3| = -x+3$
$|y-3| \le 0$ when $-1 \le y \le 3$, $|y-3| = -y+3$

Now under the conditions $-1 \le y \le 3$ and $-6 \le x \le -3$,

$$x+6+y+1 = -x-3-y+3$$
$$x+y+7 = -x-y$$
$$y = -x - 7/2$$

When we make our table of values we see:

x	y
-6	5/2
-5	3/2
-4	½
-3	-1/2

<u>Case II</u>: $-1 \le y \le 3$ and $x < -6$

Since $|x+6| < 0$ when $x < -6$, $|x+6| = -x-6$
$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$
$|x+3| < 0$ when $x < -6$, $|x+3| = -x-3$
$|y-3| \le 0$ when $-1 \le y \le 3$, $|y-3| = -y+3$

Now under the conditions $-1 \le y \le 3$ and $x < -6$,

$$-x-6+y+1 = -x-3-y+3$$
$$-x+y-5 = -x-y$$
$$y = 5/2$$

When we make our table of values we see:

x	y
-7	5/2
-8	5/2
-9	5/2
-10	5/2
.	.
.	.
.	.

<u>Case III</u>: $-1 \le y \le 3$ and $x > -3$

Since $|x+6| > 0$ when $x > -3$, $|x+6| = x+6$
$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$
$|x+3| \ge 0$ when $x > -3$, $|x+3| = x+3$

$|y-3| < 0$ when $-1 \le y \le 3$, $|y-3| = -y + 3$

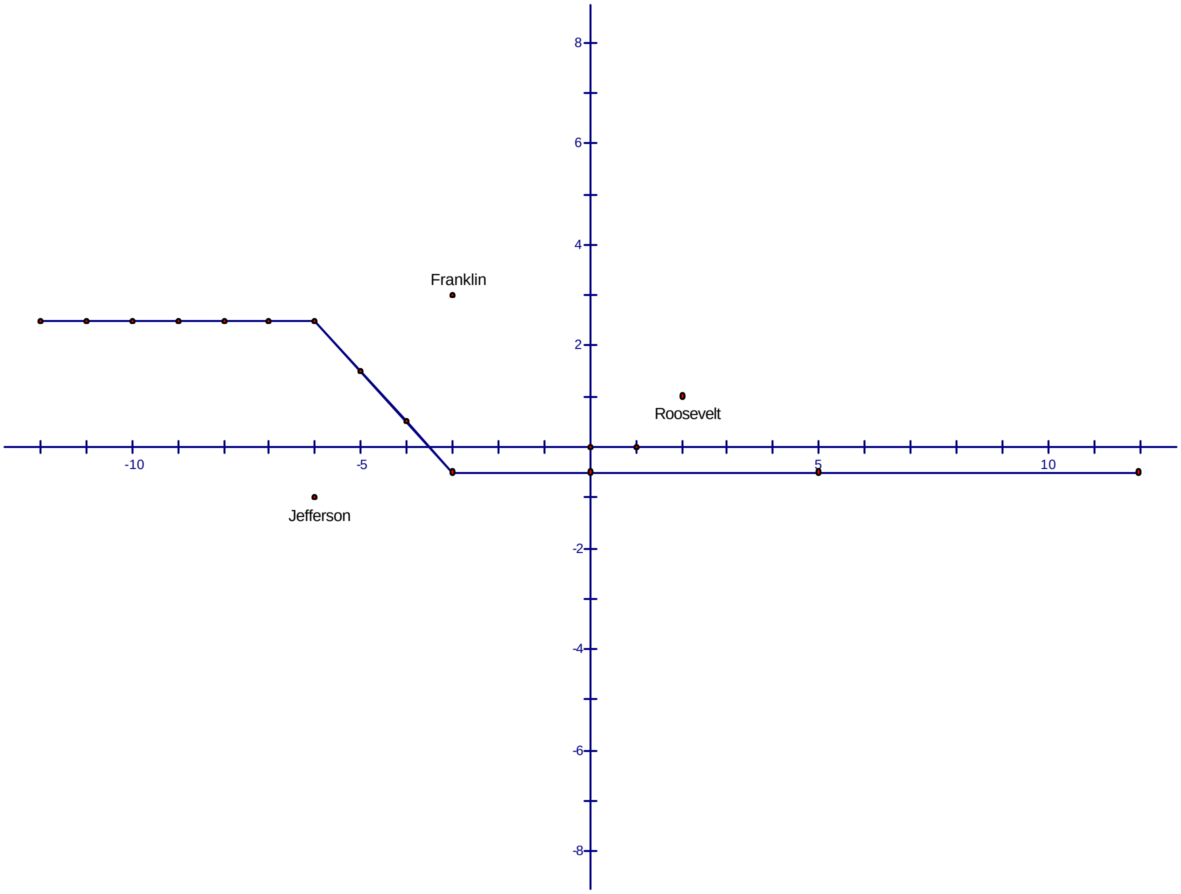
Now under the conditions $-1 \le y \le 3$ and $x > -3$,

$$x + 6 + y + 1 = x + 3 - y + 3$$
$$x + y + 7 = x - y + 6$$
$$2y = -1$$
$$y = -1/2$$

When we make our table of values we see:

x	y
-3	-1/2
-2	-1/2
-1	-1/2
0	-1/2
.	.
.	.
.	.

Below I have plotted the points and lines so far that follow d(Jefferson) =d(Franklin)



Cases IV-IX: No solutions exist when we look at these cases. You may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant III and IV, obviously any one living in the outer area will be closest to Jefferson school. Also looking in quadrants I and II, anyone living in these outer boundaries will be closer to Franklin school.

Section 2:

We will now look at the boundary between Franklin and Roosevelt school. Again we want to find the points that are equal distance from the two schools to create our boundary. We are looking for :

$$d(\text{Franklin}) = d(\text{Roosevelt})$$
$$d[(-3,3),(x,y)] = d[(2,1),(x,y)]$$
$$|x+3| + |y-3| = |x-2| + |y-1|$$

Again there are going to be different cases we need to look at., these cases are:

	$1 \le y \le 3$	$y < 1$	$y > 3$
$-3 \le x \le 2$			
$x > 2$			
$x < -3$			

<u>CaseI</u>: $1 \le y \le 3$ and $-3 \le x \le 2$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$

$|y-3| \le 0$ when $1 \le y \le 3$, $|y-3| = -y+3$

$|x-2| \le 0$ when $-3 \le x \le 2$, $|x-2| = -x+2$

$|y-1| \ge 0$ when $1 \le y \le 3$, $|y-1| = y-1$

Under the conditions $1 \le y \le 3$ and $-3 \le x \le 2$

$$x + 3 - y + 3 = -x + 2 + y - 1$$
$$x - y + 6 = -x + y + 1$$
$$-2y = -2x - 5$$
$$y = x + 5/2$$

plug into a table of values:

x	y
0	5/2
-1	3/2

Case II: $x<-3$ and $y<1$

Since $|x+3| < 0$ when $x<-3$, $|x+3| = -x-3$

$|y-3| < 0$ when $y<1$, $|y-3| = -y+3$

$|x-2| < 0$ when $x<-3$, $|x-2| = -x+2$

$|y-1| < 0$ when $y<1$, $|y-1| = -y+1$

Under the conditions $x<-3$ and $y<1$,

$$-x - 3 - y + 3 = -x + 2 - y + 1$$
$$-x - y = -x - y + 3$$

but $0 \ne 3$ so there is no solution for $x<-3$ and $y<1$

Case III: $-3 \le x \le 2$ and $y<1$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$

$|y-3| <0$ when $y<1$, $|y-3| = -y+3$

$|x-2| \le 0$ when $x<-3$, $|x-2| = -x+2$

$|y-1| <0$ when $y<1$, $|y-1| = -y+1$

Under the conditions $-3 \le x \le 2$ and $y<1$,

$$x + 3 – y + 3 = -x + 2 – y + 1$$

$$x – y +6 = -x – y +3$$

$$2x = -3$$

$$x = -3/2$$

Now plug into a table of values:

x	y
-3/2	1
-3/2	0
-3/2	-1
-3/2	-2
.	.
.	.
.	.

Case IV: $-3 \le x \le 2$ and $y \ge 3$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$

$|y-3| \ge 0$ when $y \ge 3$, $|y-3| = y-3$

$|x-2| \le 0$ when $x<-3$, $|x-2| = -x+2$

$|y-1| >0$ when $y \ge 3$, $|y-1| = y-1$

Under the conditions $-3 \le x \le 2$ and $y \ge 3$,

$$x + 3 + y – 3 = -x + 2 + y – 1$$

$$x + y = -x + y +1$$

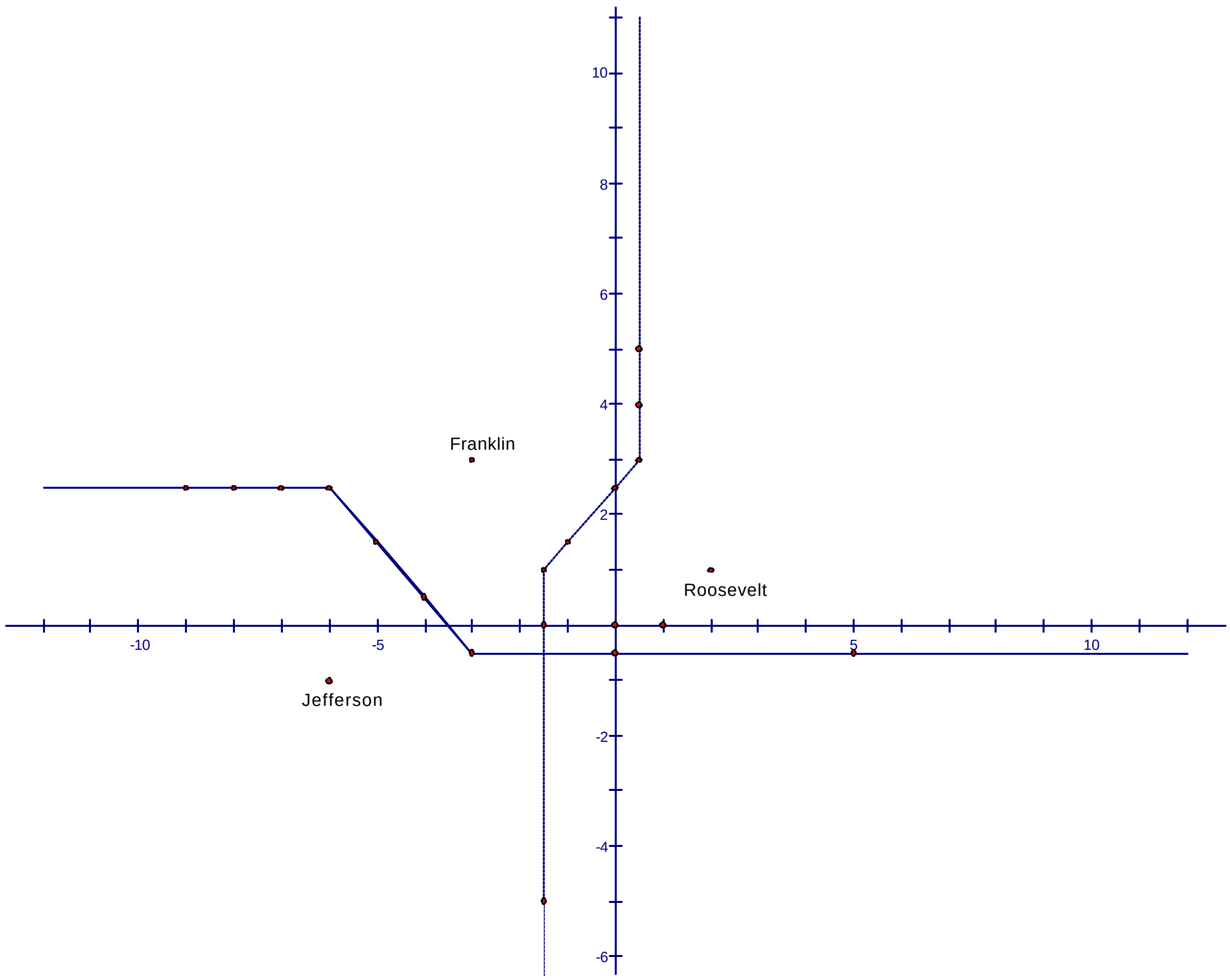
$$2x = 1$$

$$x = ½$$

plug into a table of values

x	y
½	3
½	4
½	5
.	.
.	.
.	.

Below I have plotted the points and lines so far that follow d(Roosevelt) = d(Franklin) with a dotted line.



<u>Cases V-IX</u>: No solutions exist when we look at these cases. Again you may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant I and IV, obviously any one living in the outer area will be closest to Roosevelt school. Also looking in quadrant II and III, anyone living in these outer boundaries will be closer to Franklin school.

Section 3

Lets now focus on a boundary between Roosevelt and Jefferson school. The boundary line needs to fall on the points were the distance between Jefferson and Roosevelt are the same. So we need :

$$d(\text{Jefferson}) = d(\text{Roosevelt})$$
$$d[(-6,-1),(x,y)] = d[(2,1),(x,y)]$$
$$|x+6| + |y+1| = |x-2| + |y-1|$$

Again there are going to be different cases we need to look at., these cases are:

	$-1 \le y \le 1$	$y <-1$	$y > 1$
$-6 \le x \le 2$			
$x > 2$			
$x < -6$			

<u>Case I:</u> $-6 \le x \le 2$ and $y < -1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$
$|y+1| < 0$ when $y < -1$, $|y+1| = -y +1$
$|x-2| \le 0$ when $-6 \le x \le 2$, $|x-2| = -x +2$
$|y-1| < 0$ when $y < -1$, $|y-1| = -y +1$

Now under the conditions $-6 \le x \le 2$ and $y < -1$,

$$x + 6 - y - 1 = - x + 2 - y + 1$$
$$x - y + 5 = - x - y + 3$$
$$2x = -2$$
$$x = -1$$

plug into a table of values:

x	y
-1	-2
-1	-3
-1	-4
.	.
.	.
.	.

<u>Case II</u>: $-6 \le x \le 2$ and $-1\le y \le 1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$
$|y+1| \ge 0$ when $-1< y \le -1$, $|y+1| = y+1$
$|x-2| \le 0$ when $-6 \le x \le 2$, $|x-2| = -x +2$
$|y-1| \le 0$ when $-1 \le y \le -1$, $|y-1| = -y +1$

Now under the conditions $-6 \le x \le 2$ and $-1 \le y \le -1$,

$$x + 6 +y +1 = -x + 2 -y +1$$
$$x + y + 7 = -x - y +3$$
$$x + y = -x - y - 4$$
$$y = -x - 2$$

plug into a table of values

x	y
-1	-1
-2	0
-3	1

<u>Case III:</u> $-6 \le x \le 2$ and $y > 1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$

$|y+1| > 0$ when $y > 1$, $|y+1| = y+1$
$|x-2| \leq 0$ when $-6 \leq x \leq 2$, $|x-2| = -x+2$
$|y-1| > 0$ when $y > 1$, $|y-1| = y-1$

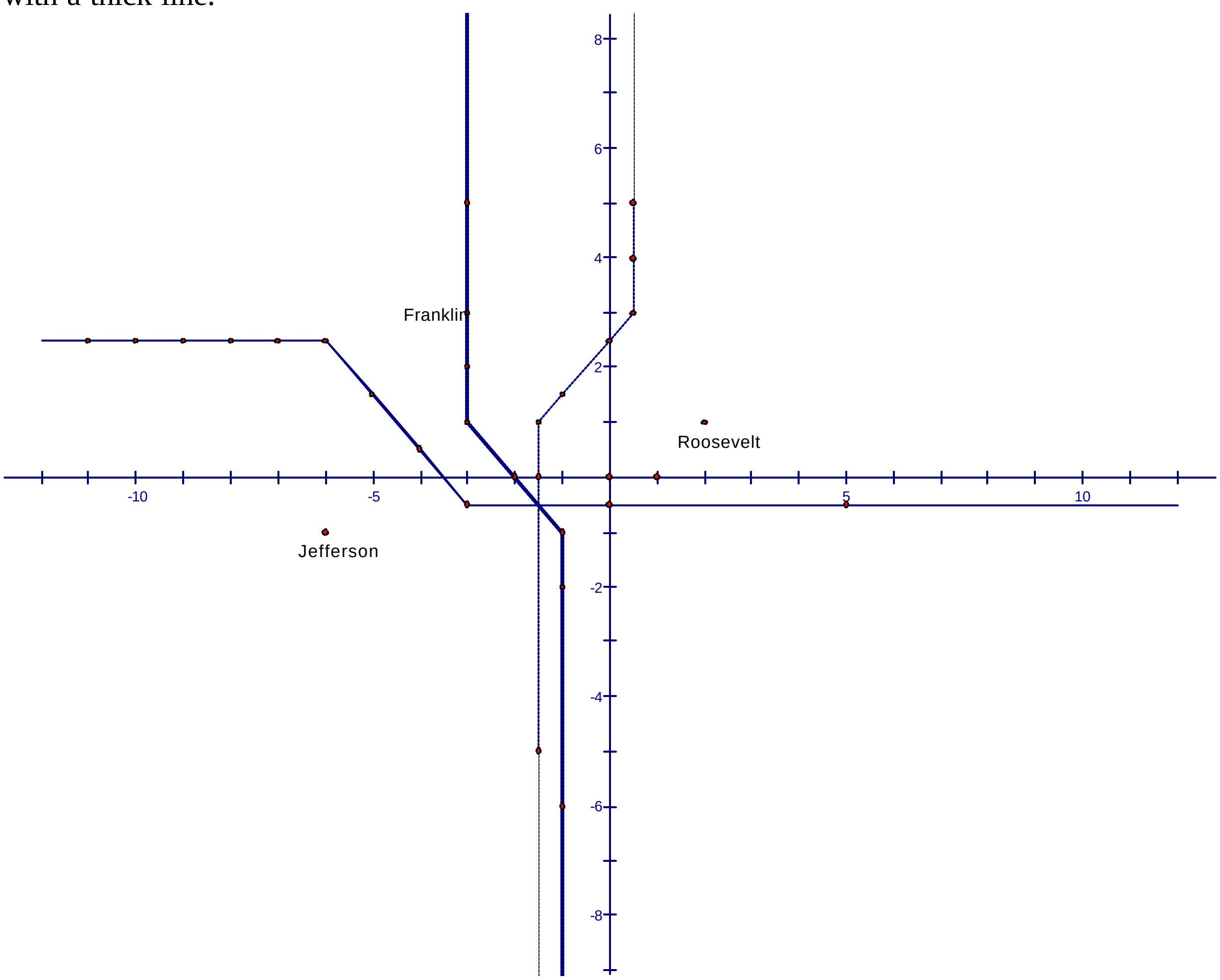
Now under the conditions $-6 \leq x \leq 2$ and $1 < y$,

$$x + 6 + y + 1 = -x + 2 + y - 1$$
$$x + y + 7 = -x + y + 1$$
$$2x = -6$$
$$x = -3$$

plug into a table of values:

x	y
-3	2
-3	3
-3	4
.	.
.	.
.	.

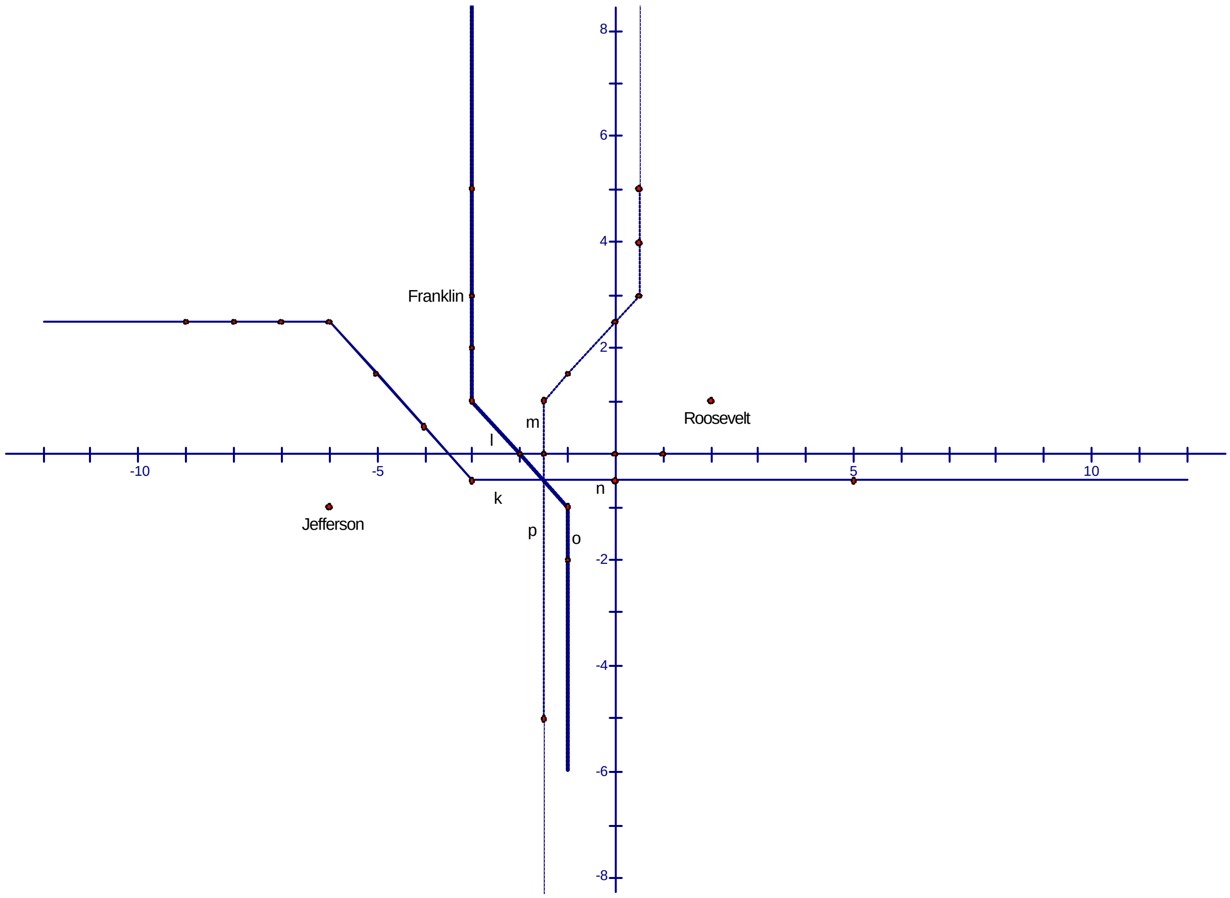
Below I have plotted the points and lines so far that follow d(Roosevelt) = d(Jefferson) with a thick line.



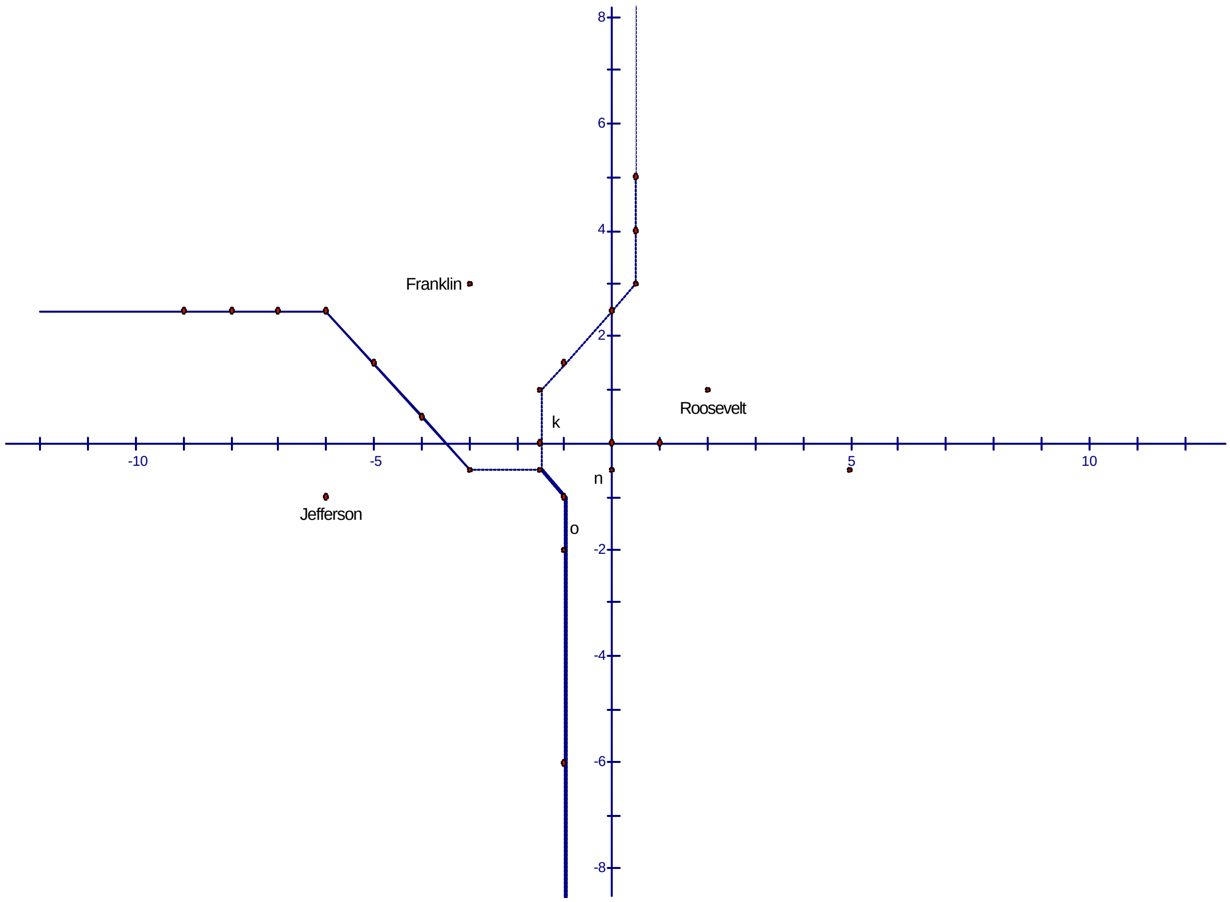
Cases IV-IX: No solutions exist when we look at these cases. Again you may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant I and IV, obviously any one living in the outer area will be closest to Roosevelt school. Also looking in quadrant II and III, anyone living in these outer boundaries will be closer to Jefferson school.

Section 4

Now we need to look at the information and use some basic logic to interpret the information we have. The previous figure has all the boundaries between two school districts on it. We must first find the point were the three boundaries intersect. This is at (-1/2,-1/2). Now we will look at each boundary from this point. I will label the boundaries for ease of explanation as follows:



First look at boundary *k*, it is obviously a correct boundary because there is no other boundary close that would properly divide Franklin and Jefferson. Boundary *l* was intended to divide Jefferson and Roosevelt, this boundary is not necessary because the points on this line are closest to Franklin. Since we removed *l*, *m* must stay to maintain a boundary between Roosevelt and Franklin. Now boundary *n* is not necessary because it is the boundary between Jefferson and Franklin. Roosevelt school is closer, so we can remove *n*. Our boundary *p* is for Franklin and Roosevelt. But we are closer to Jefferson and Roosevelt so we will disregard this boundary. Now *o* is left. We will keep this boundary since *p* was removed, *o* must stay to maintain the boundary between Jefferson and Roosevelt. After removing some of the boundaries, we have our final solution to a school district such that everyone is attending the school they live closest to. Below is the graph of this solution:



Problem Three

The telephone company wants to set up pay phone booths so that everyone living with in 12 blocks of the center of town is with in four blocks of a payphone booth. Money is tight, the telephone company wants to put in the least amount of booths possible such that this is true.

Taxicab geometry is the logical choice for solving this problem. For the people can not walk through backyards or jump over buildings to use the phone. They must stick to the street.

We must break this problem down into different sections.

Section 1:

We must first draw the boundary lines that are with in 12 blocks of the center of town. To do this we will need to find the areas that are ≤ 12 blocks. We need to look at the lines that are a distance of 12 from the center of town. To do this look at the equation:

$d[(0,0),(x,y)] = 12$

$|x - 0| + |y-0| = 12$

As in the previous problem, we must evaluate this formula with different cases because of the absolute values. We must look at the following cases:

	$-12 = x \leq 0$	$12 = x \geq 0$
$-12 = y \leq 0$		
$12 = y \geq 0$		

Case I: $12 = x \geq 0$ and $12 = y \geq 0$
since $|x-0| \geq 0$ when $12 = x \geq 0$, $|x-0| = x-0$
$|y-0| = 0$ when $12 = y \geq 0$, $|y-0| = y-0$

when $12 = x \geq 0$ and $12 = y \geq 0$,
$x-0 + y-0 = 12$
$y = -x + 12$

Case II: $12 = x \geq 0$ and $-12 = y \leq 0$
since $|x-0| \geq 0$ when $12 = x \geq 0$, $|x-0| = x-0$
$|y-0| = 0$ when $-12 = y \leq 0$, $|y-0| = -y+0$

when $12 = x \geq 0$ and $-12 = y \leq 0$
$x - 0 - y + 0 = 12$
$x - 12 = y$

Case III: $-12 = x \leq 0$ and $-12 = y \leq 0$
since $|x-0| = 0$ when $-12 = x \leq 0$, $|x-0| = -x+0$
$|y-0| = 0$ when $-12 = y \leq 0$, $|y-0| = -y+0$

when $-12 = x \leq 0$ and $-12 = y \leq 0$,
$- x + 0 - y + 0 = 12$
$- x -12 = y$

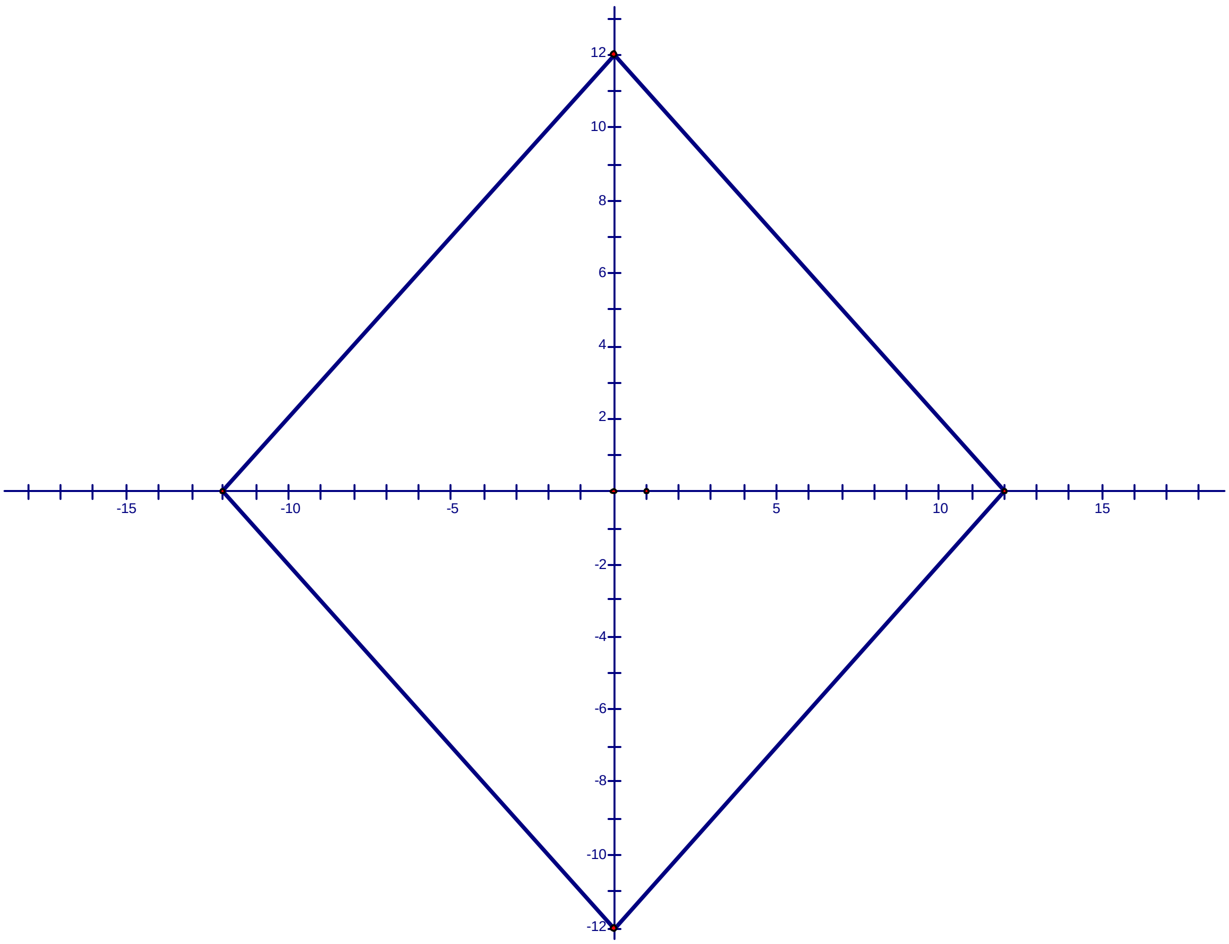
Case IV: $-12 = x \leq 0$ and $12 = y \geq 0$,

since $| x-0 | = 0$ when $-12 = x \leq 0$, $| x-0 | = -x+0$
$| y-0 | = 0$ when $12 = y \geq 0$, $| y-0 | = y-0$

when $-12 = x \leq 0$ and $12 = y \geq 0$,

$$-x + 0 + y - 0 = 12$$
$$y = x + 12$$

We have found four lines, each in a different quadrant, that have a distance of 12 to the origin. Below I have graphed our four cases to see the boundary were everyone with in these lines are within twelve blocks of down town.



What we have done here is graphed a circle of radius 12 in taxicab geometry.

Section 2

Now we need to find an equation for a line such that every one living on the boundary in quadrant I will be with in four blocks of. We can find this line by using the equation

$$d[(x,-x-12),(x_1,y_1)] = 4$$
$$| x – x_1 + (-x+12) – y_1 = 4$$

we only need to look at the cases were we are with in our original boundary. Therefore $x_1 = x$ and $y_1 = y$. Now $x – x_1 = x – x_1$ and $(-x+12) – y_1 = -x +12 – y_1$

$$x – x_1 + -x + 12 – y_1 = 4$$
$$-x_1 + 8 = y_1$$

lets call this line *l*

The equation $-x_1 + 8 = y_1$ gives us a line to place possible payphones that is the optimal distance from our outer boundary in quadrant I. Everything between line *l* and our outer boundary will be = 4 blocks of our line *l*. We now need to find the boundary line for everyone that can use a phone located on line *l* that lives below it. To do this we will look at the equation :

$$d[(x_1,x_1-8),(x_2,y_2)] = 4$$
$$|x_2-x_1| + |y_2-(x_1-8)| = 4$$

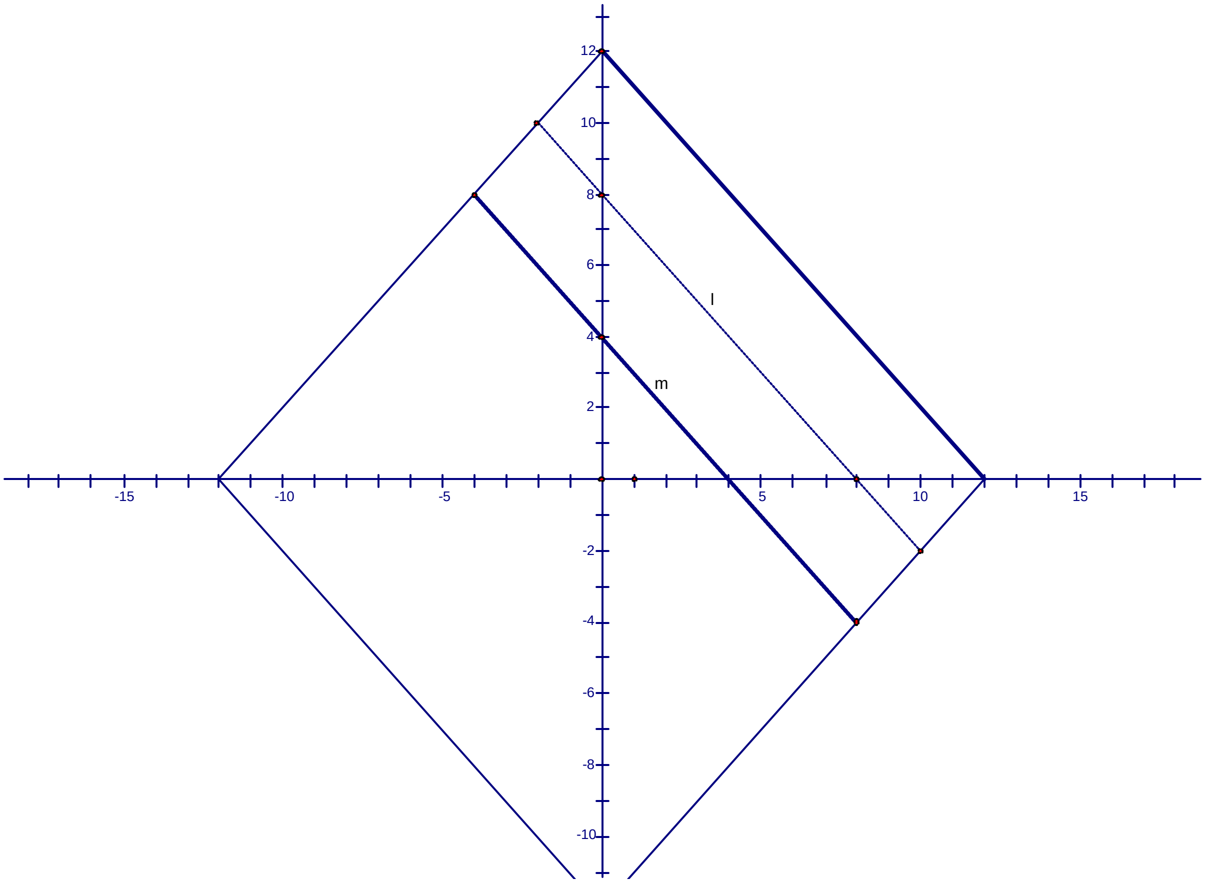
Again, we only need to look at the cases were we are with in our boundary. Therefore $x_2 = x_1$ and $y_2 = y_1$. Now $x_2 – x_1 = -x_2 +x_1$ and $y_2-(-x_1+8) = -y_2+(-x_1 +8)$

Now our equation becomes:

$$-x_2 +x_1-y_2+(-x_1 +8) = 4$$
$$-x_2 -y_2+8 = 4$$
$$-x_2 + 4 = y_2$$

lets call this line *m*

Now graph this equation to see the boundaries of our possible placement of phones. Everyone living between the two thick lines will be with in four blocks of phones located on the dotted line.



As we did before, we must find a line below line such that people living on or below this boundary (line *m*) will be with in four blocks. To do this we will use the equation:

$$d[(x_2,-x_2+4),(x_3,y_3)] = 4$$

Again, we only need to look at the cases were we are with in our boundary. Therefore $x_3 = x_2$ and $y_3 = y_2$. Now $x_3 - x_2 = -x_3 + x_2$ and $y_3-(-x_2+4) = -y_3+(-x_2+4)$
Now our equation becomes:

$$-x_3 + x_2 - y_3 + (-x_2 + 4) = 4$$
$$-x_3 = y_3$$

lets call this line *n*

If phones are placed along this line, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

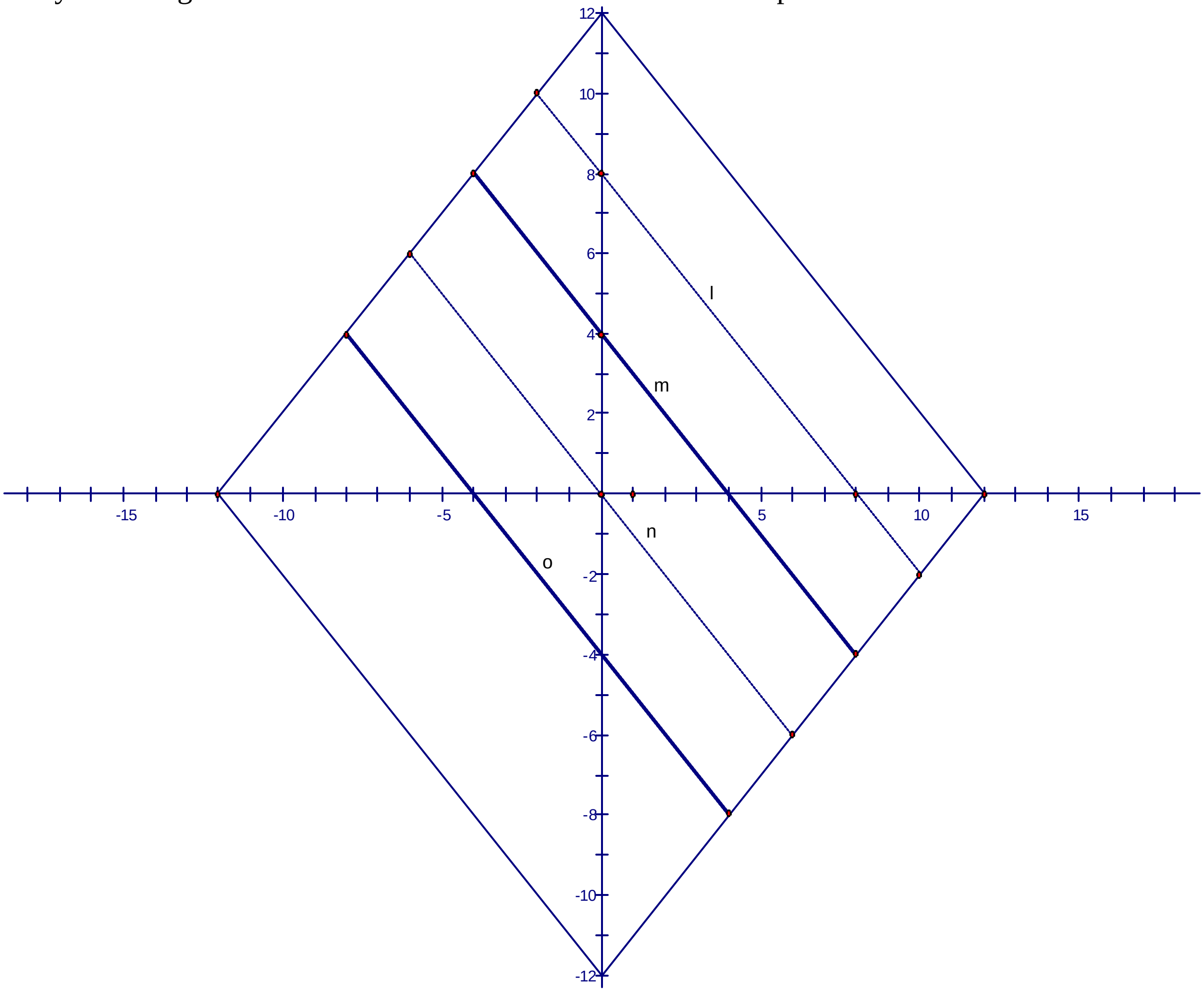
$$d[(x_3,-x_3),(x_4,\ y_4)] = 4$$

Again, we only need to look at the cases were we are with in our boundary. Therefore $x_4 = x_3$ and $y_4 = y_3$. Now $x_4 - x_3 = -x_4 + x_3$ and $y_4-(-x_3) = -y_4+(-x_3)$
Now our equation becomes:

$$-x_4 + x_3 - y_4 + (-x_3) = 4$$
$$-x_4 - 4 = y_4$$

lets call this line *o*

When we graph these two lines we will see the boundaries (lines *m* and *o*) such that everyone living between the two will be with in four blocks of phones located on line *n*.



This process needs to be repeated one more time. As we did before, we must find a line below line *o* such that people living on or below this boundary will be with in four blocks of a phone. To do this we will use the equation:

$$d[(x_4,-x_4-4),(x_5, y_5)] = 4$$

We only need to look at the cases were we are with in our boundary. Therefore
$x_5 = x_4$ and $y_5 = y_4$. Now $x_5 – x_4 = -x_5 +x_4$ and $y_5-(-x_4-4) = -y_5+(-x_4-4)$
Now our equation becomes:

$$-x_5 +x_4 –y_5+(-x_4-4) = 4$$
$$-x_5-8 = y_5$$

Call this line *p*

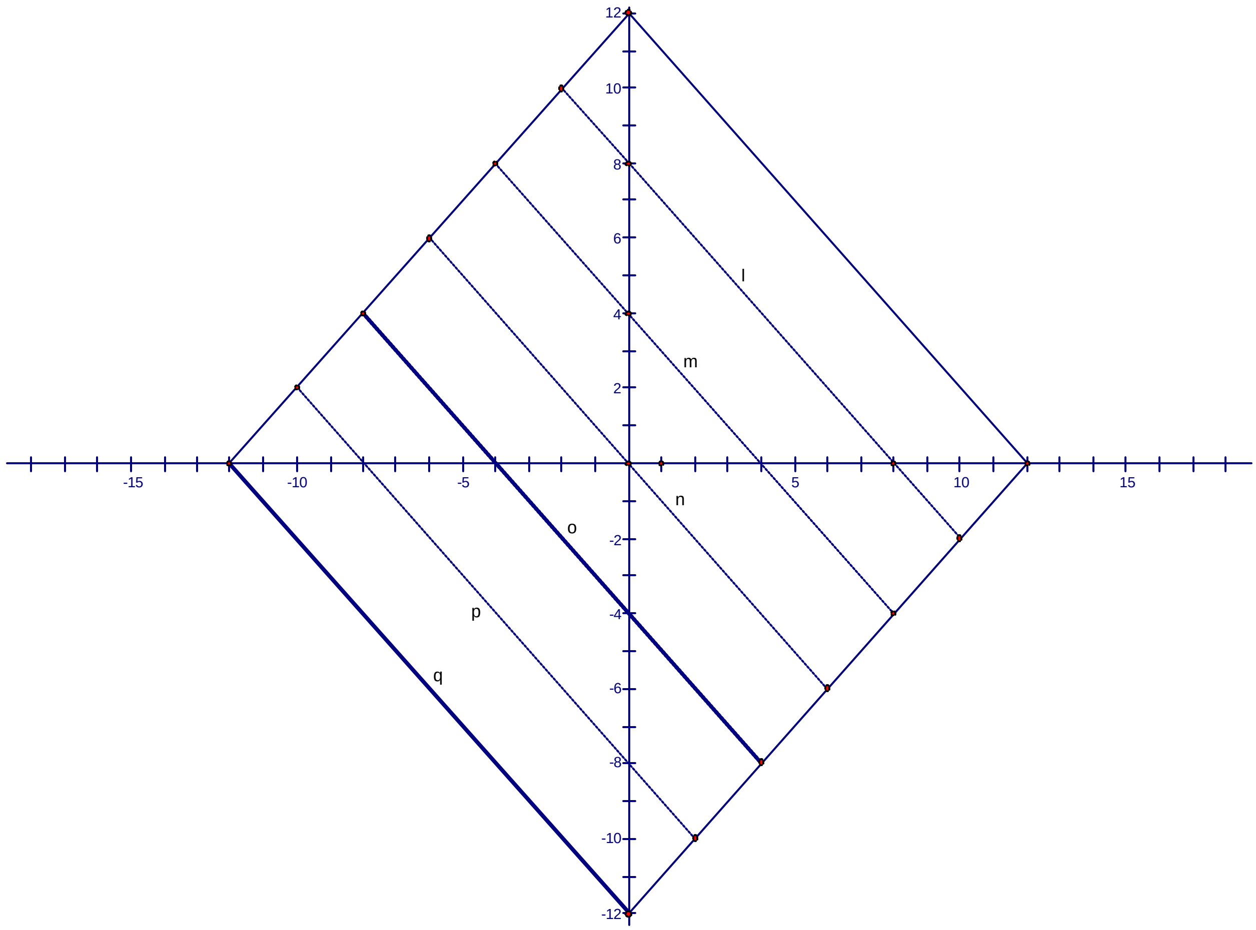
If phones are placed along this line *p*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

$$d[(x_5,-x_5-8),(x_6,y_6)] = 4$$

We only need to look at the cases were we are with in our boundary. Therefore
$x_6 = x_5$ and $y_6 = y_5$. Now $x_6 – x_5 = -x_6 +x_5$ and $y_6-(-x_5-8) = -y_6+(-x_5-8)$
Now our equation becomes:

$$-x_6 +x_5 –y_6+(-x-8) = 4$$
$$-x_6-12 = y_6$$

lets call this line *q*
Line *q* is the same line as our outer boundary in quadrant III. Below I have graphed the boundaries and the line that the phones can be placed on.



We have found optimal lines to put our phone booths on, but we do not know were on these lines to place them. We will need to find out more information.

Section 3

We will repeat the exact same process as in section two. This time we will start with finding the line such that everyone living on the boundary in quadrant II is with in four blocks of. From section 1, we know the equation for our outer boundary is $y = x_7+12$
We need to solve the equation:

$$d[(x_7,x_7+12),(x_8,y_8)] = 4$$

For all the solutions to the upcoming problems in this section $x_n \geq x_{n+1}$ and $y_n \leq y_{n+1}$ because we only need to work within our stated boundaries. So the lines that we will be solving will always have x values greater than the previous line and y values less then the previous line. Therefore all

$d[(x_n,y_n),(x_{n+1},y_{n+1})] = 4,$

$|x_{n+1}-x_n| \leq 0$ when $x_n \leq x_{n+1}$, so $|x_{n+1}-x_n| = x_{n+1}-x_n$

$|y_{n+1}-y_n| \geq 0$ when $y_n \geq y_{n+1}$, so $|y_{n+1}-y_n| = -y_{n+1}+y_n$

Back to solving $d[(x_7,x_7+12),(x_8,y_8)] = 4,$

$$x_8-x_7-y_8+(x_7+12) = 4$$

$$x_8+8 = y_8$$

call this line *r*

If phones are placed along this line *r*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

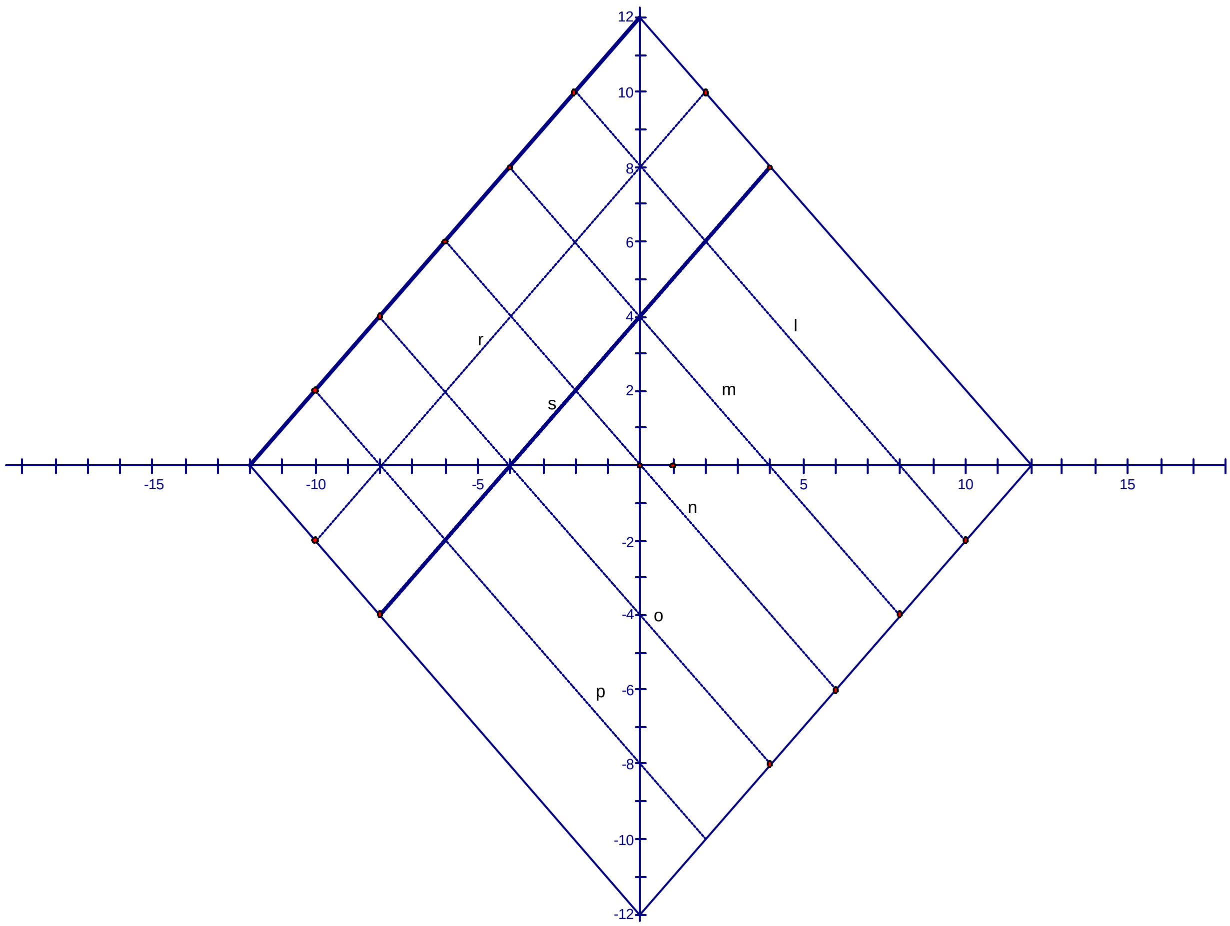
$$d[(x_8,x_8+8),(x_9,y_9)] = 4$$

$$x_9-x_8-y_9+(x_8+8) = 4$$

$$x_9+4 = y_9$$

call this line *s*

When we graph these two lines we will see that the lines *s* and $y = x_7+12$ (our outer boundary) are boundaries such that everyone living between the two will be with in four blocks of phones located on line *r*. I have graphed this below.



As we did before, we must find a line below line such that people living on or below this boundary (line *s*) will be with in four blocks. To do this we will use the equation:

$$d[(x_9,x_9+4),(x_{10},y_{10})] = 4$$
$$x_{10}-x_9-y_{10}+(x_9+4) = 4$$
$$x_{10} = y_{10}$$

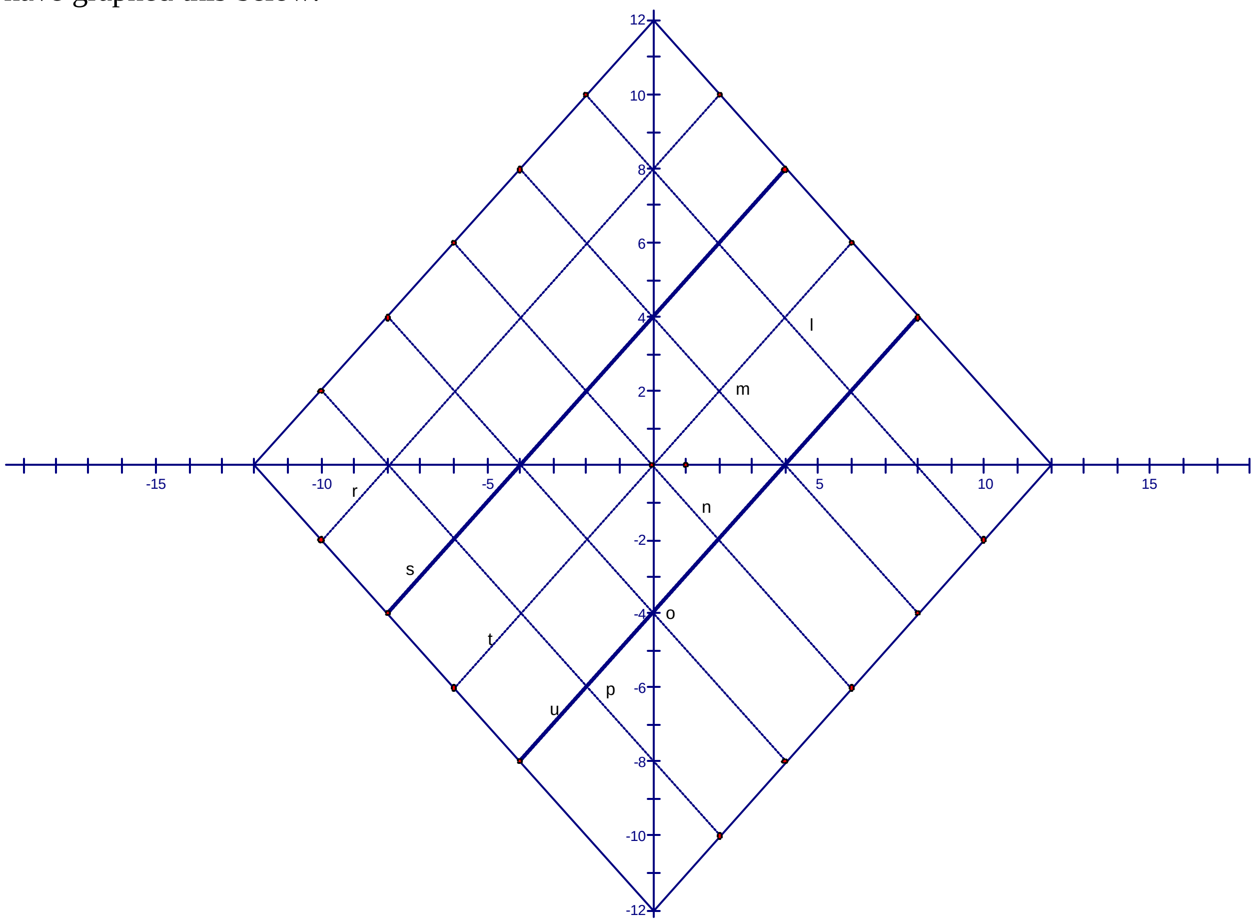
call this line *t*

If phones are placed along this line *t*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

$$d[(x_{10},x_{10}),(x_{11},y_{11})] = 4$$
$$x_{11}-x_{10}-y_{11}+x_{10} = 4$$
$$x_{11}-4 = y_{11}$$

call this line *u*

When we graph these two lines we will see that the lines *r* and *u* are boundaries such that everyone living between the two will be with in four blocks of phones located on line *t*. I have graphed this below.



This process needs to be repeated one more time. As we did before, we must find a line below line *u* such that people living on or below this boundary will be with in four blocks of a phone. To do this we will use the equation:

$$d[(x_{11},x_{11}-4),(x_{12},y_{12})] = 4$$

$$x_{12}-x_{11}-y_{12}+(x_{11}-4) = 4$$

$$x_{12}-8 = y_{12}$$

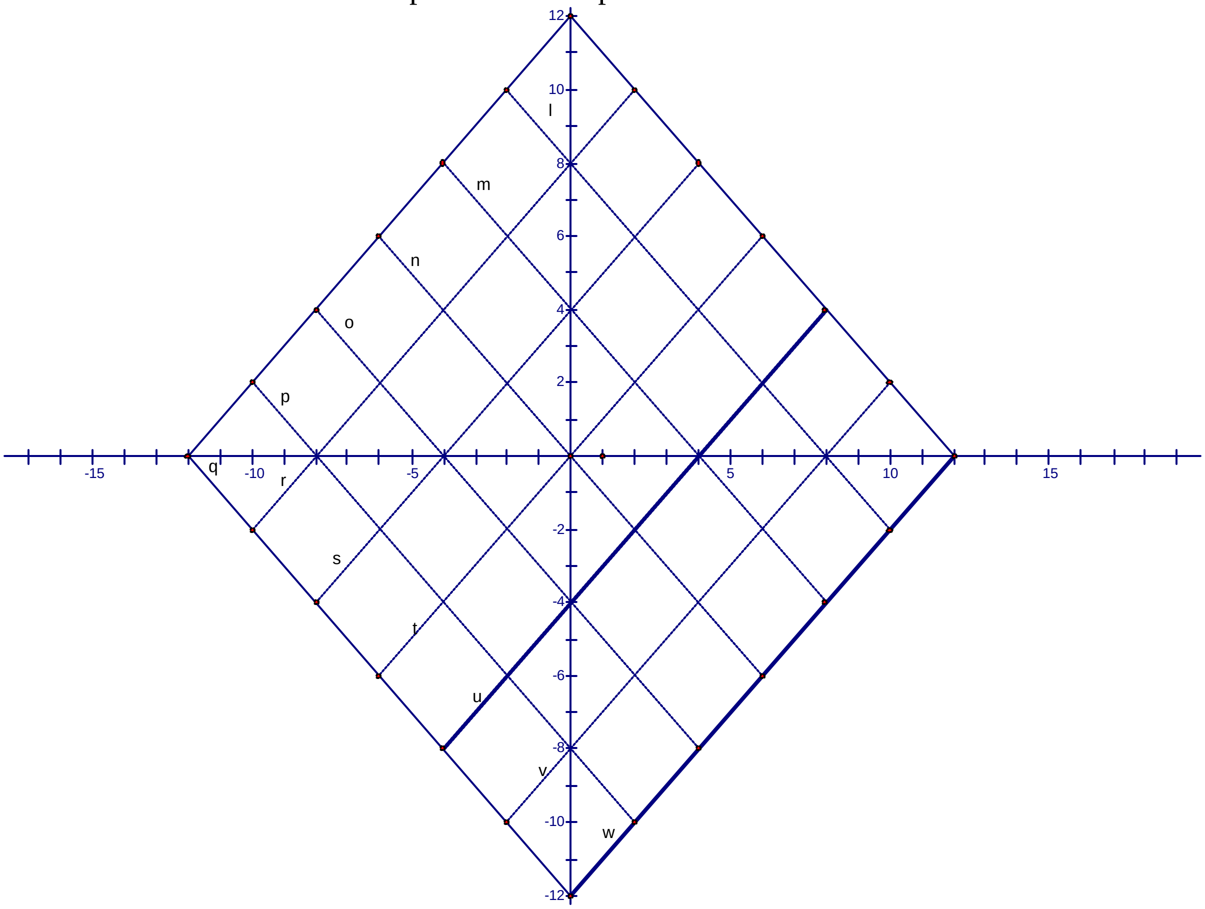
call this line *v*

If phones are placed along this line *v*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

$$d[(x_{12},x_{12}-4),(x_{13},y_{13})] = 4$$
$$x_{13}-x_{12}-y_{13}+(x_{12}-8) = 4$$
$$x_{13}-12 = y_{13}$$

lets call this line *w*

Line *w* is the same line as our outer boundary in quadrant IV. Below I have graphed the boundaries and the line that the phones can be placed on.



Section 4

Now we need to simply interpret our graph. The dotted lines in our graph above represent optimal places to put our phone booths. Until now we have not had a specific location, only a line for the booth to be placed on. Notice that the dotted lines intersect. These are the points were phone booths should be located. You can see that all the work we have done has divided our total area into 9 sections. Each one of these 9 sections as an intersection. By the way we constructed our graph, we know these intersections are the optimal places to put the least amount of phone booths such that everyone living with in 12 blocks of the center of town is with in 4 blocks of a pay phone.

We can see the taxicab geometry is a very useful model of urban geography. Only a pigeon would benefit from the knowledge that the distance between two buildings on opposite ends of a city is a strait line. For people, taxicab distance is the "real" distance. Taxicab geometry has many applications and is relatively easy to explore. I challenge the reader to explore other ideas in taxicab geometry. What do familiar geometric figures look like in taxicab geometry. We have already seen that circles in taxicab geometry look like squares. Are other figures transmuted?

References

Gardner, M. (1997). *The Last Recreations.* Springer Verlag, New York.
Golland, L. (1990). Karl Menger and Taxicab Geometry. *Mathematics Magazine*, 63 (5), 326-327.
Krause, E. F. (1975). *Taxicab Geometry*. Dover Publications, New York.
Reynolds, B.E. (1980). Taxicab Geometry. *Pi Mu Epsilon Journal*, 7, 77-88.

