Erratum: An Improvement on the article *Taxi Cab Geometry: History and Applications*, TMME, Vol2, no.1, p. 38 – 64

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Abstract: *As a high school student from Germany I did a Mathematics research paper entitled Taxicab Geometry: Fundamentals and Applications. During my research I found the article Taxicab Geometry: History and Applications by Chip Reinhardt which was published in the edition Vol.2, no. 1 (p. 38 – 64) of this journal. The article was very helpful for my coursework and I'd like to compliment the author and everyone who is involved in the journal on their work. In my coursework I created an application example similar to the one Chip Reinhardt used in his article. I solved my example in the same way Mr. Reinhardt solved his, but I made some improvements. In his article there were some errors in calculations which I revised. In the following article I'd like to suggest a rectified solution to Chip Reinhardt's example.*

1. Creation of school districts

In New York school district boundaries should be created such that every student is going to the closest school.

The application of Taxicab geometry to solve this problem is a logical choice as the students have to stick to the streets on their way to school.

There are three schools in between which the school boundaries need to be found: Franklin at (-3 / 3), Jefferson at (-6 / 1) and Roosevelt at (2 / 1). In a co-ordinate system the schools are located as following:

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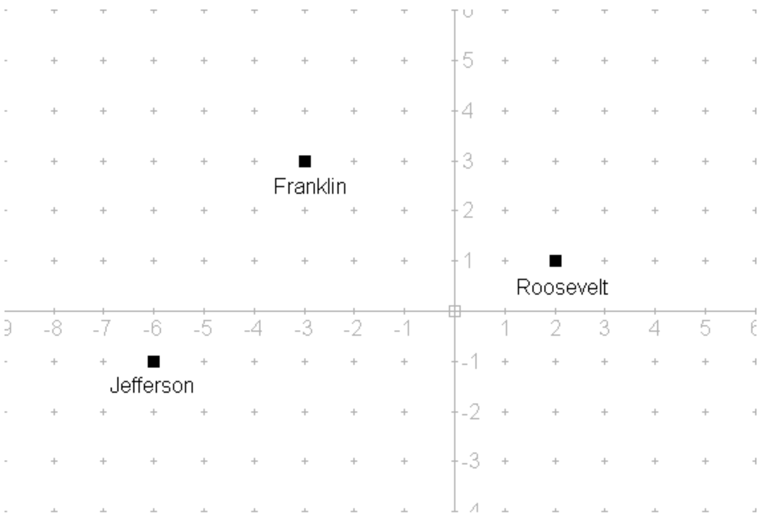


To solve the whole problem we need to break the solution down into different sections.

Section 1

In the first section you focus on the boundary between Franklin and Jefferson school. To find the boundary line between these two schools you need to find the points which are equal distance from Franklin and Jefferson school. Expressed in an equation, which will be named a_1 in the following calculation, you need:

$$d(Jefferson) = d(Franklin)$$
$$d\left[(-6,-1),(x,y)\right] = d\left[(-3,3),(x,y)\right]$$
$$|-6-x|+|-1-y|=|-3-x|+|y-3|$$
$$|x+6|+|y+1|=|x+3|+|y-3|$$

To solve this equation for x and y we look at nine different cases as the solution with the absolute values gets difficult. The different cases are:

	$-1 \le y \le 3$	$-1 > y$	$3 < y$
$-6 \le x \le -3$	Case I	Case IV	Case VII
$-6 > x$	Case II	Case V	Case IIX
$-3 < x$	Case III	Case VI	Case IX

Chip Reinhardt continued like this :

We look at these cases because the absolute values will change the solutions.

Case I: $-6 \leq x \leq -3$ and $-1 \leq y \leq 3$

Since $|x+6| \geq 0$, when $-6 \leq x \leq -3$, $|x+6| = x+6$

Since $|y+1| \geq, 0$ when $-1 \leq y \leq 3$, $|y+1| = y+1$

Since $|x+3| \leq, 0$ when $-6 \leq x \leq -3$, $|x+3| = -x+3$

Since $|y-3| \leq, 0$ when $-1 \leq y \leq 3$, $|y-3| = -y+3$

The underlined, red part is mathematically incorrect because $|x+3|$ and $|y-3|$ is always positive or equal to 0 so it is always ≥ 0 and

Improvement:

To avoid this mistake I chose to write the simplifications/calculations for each case like this:

Case I: Under the terms $-6 \leq x \leq -3$ and $--1 \leq y \leq 3$ you can simplify the starting equation a_1 as following:

Because $x+6 \geq 0$, when $-6 \leq x \leq -3$, you can replace $|x+6|$ by $x+6$.

Because $y+1 \geq 0$, when $-1 \leq y \leq 3$, you can replace $|y+1|$ by $y+1$

Because $x+3 \leq 0$, when $-6 \leq x \leq -3$, you can replace $|x+3|$ by $-x+3$

Because $y-3 \leq 0$, when $-1 \leq y \leq 3$, you can replace $|y-3|$ by $-y+3$

x	y
-6	5/2
-5	3/2
-4	½
-3	-1/2

If these simplifications are substituted into equation a_1 you can observe:

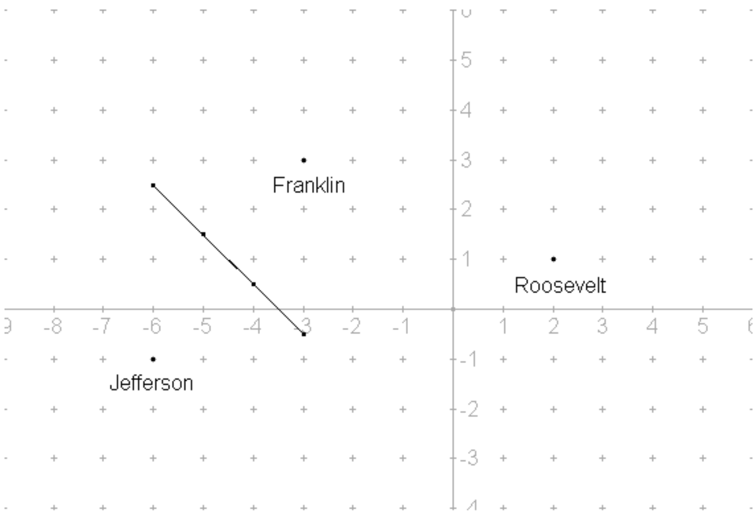
$$x+6+y+1=-x-3-y+3$$

$$x+y+7=-x-y \qquad |-x \ +y\ /2$$

$$y=-x-7/2$$

To be able to draw the boundaries easier in at the end of the calculation you put your values into a table.

Drawn into the co-ordinate system the boundary between Franklin and Jefferson in the region where $-6 \leq x \leq -3$ and $-1 \leq y \leq 3$ looks like this:



<u>Case II:</u> If –6 > x and -1 ≤ y ≤ 3 you can simplify the equation a_1 from the beginning as following:

Because $x+6<0$, when -6 > x, you can replace $|x+6|$ by $-x-6$.

Because $y+1\geq 0$, when -1 ≤ y ≤ 3, you can replace $|y+1|$ by $y+1$

Because $x+3<0$, when -6 > x, you can replace $|x+3|$ by $-x-3$

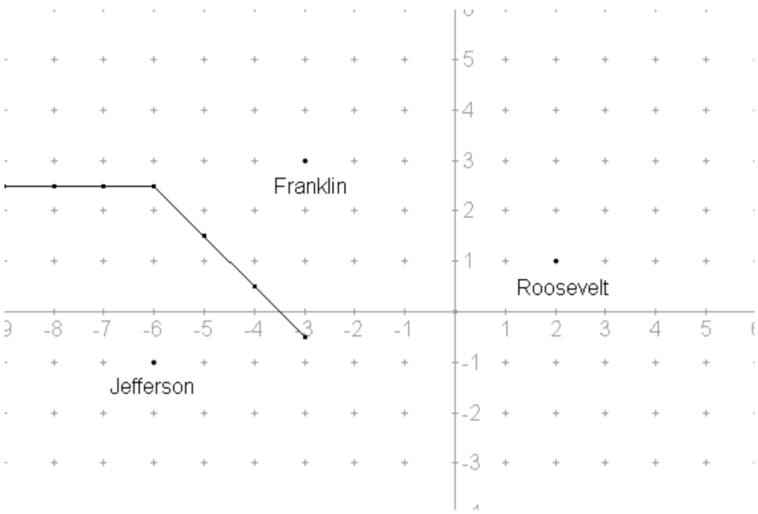
Because $y-3\leq 0$, when –1 ≤ y ≤ 3, you can replace $|y-3|$ by $-y+3$

Substituted into a_1 you can observe:

$$-x-6+y+1=-x-3-y+3$$
$$-x+y-5=-x-y$$
$$y=5/2$$

x	y
-7	5/2
-8	5/2
-9	5/2
...	5/2

In the following picture you can see the boundary between Franklin and Jefferson in the region where –6 > x and -1 ≤ y ≤ 3.



Case III: Under the conditions x > -3 and -1 ≤ y ≤ 3 you can simplify the starting equation a_1 like this:

Because $x+6>0$, when -3 < x, you can replace $|x+6|$ by $x+6$.

Because $y+1\geq 0$, when -1 ≤ y ≤ 3, you can replace $|y+1|$ by $y+1$

Because $x+3>0$, when -3 < x, you can replace $|x+3|$ by $x+3$

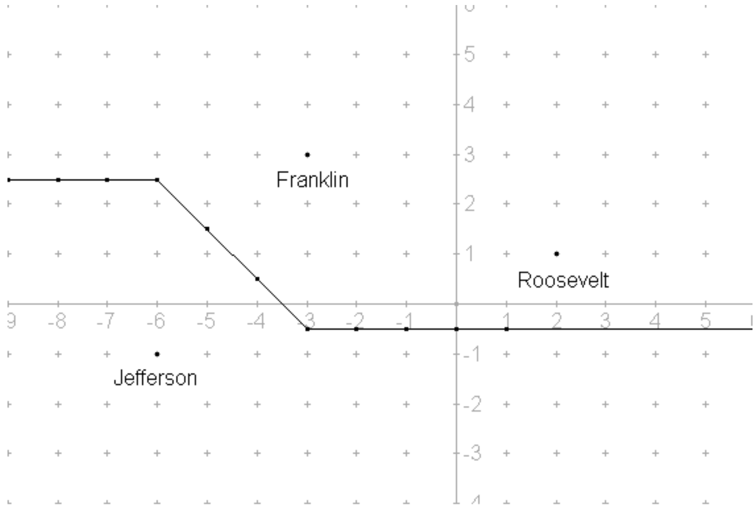
Because $y-3\leq 0$, when –1 ≤y ≤ 3, you can replace $|y-3|$ by $-y+3$

If you substitute these simplifications into the starting equation a_1 you can observe:

$$x+6+y+1=x+3-y+3$$
$$x+y+7=x-y+6$$
$$2y=-1$$
$$y=-1/2$$

x	y
-3	-1/2
-2	-1/2
-1	-1/2
...	-1/2

Blow you can see the whole boundary line, including the region x > -3 and -1 ≤ y ≤ 3, between Jefferson and Franklin at which the points and lines are equal distance from both schools: $d(Jefferson)=d(Franklin)$



<u>In case IV – IX</u> no solutions exist. You don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points/lines than the other.

<u>Section 2</u>

In section two you look at the boundary between Franklin and Roosevelt school. At first you need to look at the points which are equal distance from both schools to create a boundary. Expressed in an equation (a_2) you need:

$$d(Franklin) = d(Roosevelt)$$

$$d\left[(-3,3),(x,y)\right] = d\left[(2,1),(x,y)\right]$$

$$|x+3|+|y-3| = |x-2|+|y-1|$$

To solve this equation for x and y we divide the calculation into different cases again:

	$1 \leq y \leq 3$	$1 > y$	$3 < y$
$-3 \leq x \leq 2$	Case I	Case II	Case III

2 < x	Case IV	Case V	Case VI
-3 > x	Case VII	Case IIX	Case IX

<u>Case I:</u> If these terms apply: $-3 \leq x \leq 2$ and $1 \leq y \leq 3$, you can simply a_2 as following:

Because $x+3 \geq 0$, when $-3 \leq x \leq 2$, you can replace $|x+3|$ by $x+3$.

Because $y-3 \leq 0$, when $1 \leq y \leq 3$, you can replace $|y-3|$ by $-y+3$

Because $x-2 \leq 0$, when $-3 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$

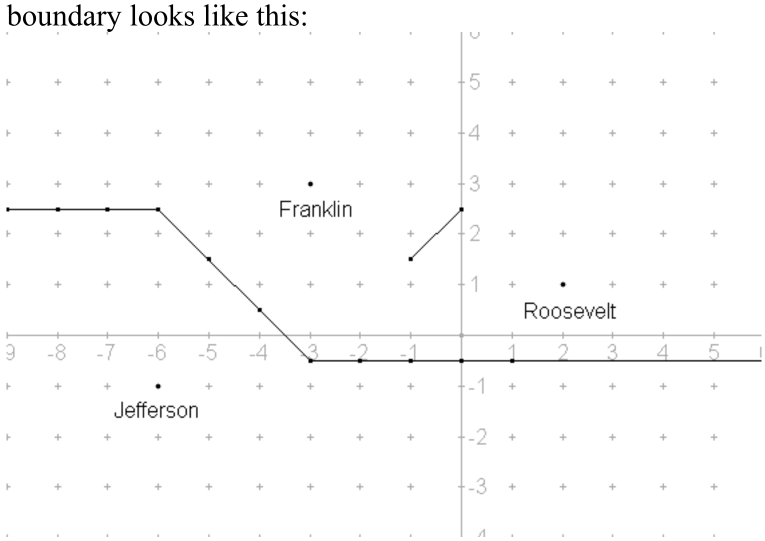
Because $y-1 \geq 0$, when $1 \leq y \leq 3$, you can replace $|y-1|$ by $y-1$

Substituted into a_2 you can simplify:

$$x+3-y+3=-x+2+y-1$$
$$x-y+6=-x+y+1$$
$$-2y=-2x-5$$
$$y=x+5/2$$

x	y
0	5/2
-1	3/2

In the table you only need to plug the values which are in the region. If x = -2; x = 1; x = 2, the condition $1 \leq y \leq 3$ doesn't apply and you don't need to plug these values. Plotted the boundary looks like this:



<u>Case II:</u> If $-3 \leq x \leq 2$ and $1 > y$ you can simplify the equation a_2 as following:

Because $x+3 \geq 0$, when $-3 \leq x \leq 2$, you can replace $|x+3|$ by $x+3$.

Because $y-3 < 0$, when $1 > y$, you can replace $|y-3|$ by $-y+3$

Because $x-2 \leq 0$, when $-3 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$

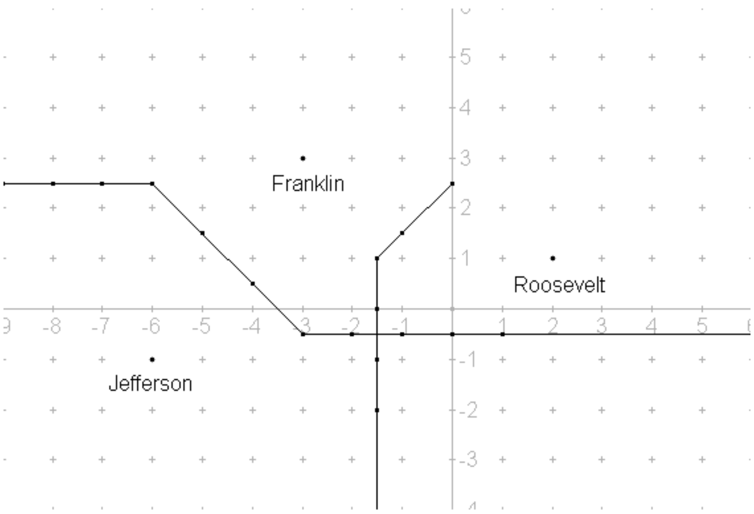
Because $y-1<0$, when 1 > y, you can replace $|y-1|$ by $-y+1$

Substituted into a_2 you can observe:

$$x+3-y+3=-x+2-y+1$$
$$x-y+6=-x-y+3$$
$$2x=-3$$
$$x=-3/2$$

x	y
-3/2	1
-3/2	0
-3/2	-1
-3/2	-2

In the following image you can see the plotted boundary in the region –3 ≤ x ≤ 2 and 1 > y:



Case III: Now we have a look at the region where –3 ≤ x ≤ 2 and 3 < y. Under these terms following simplifications for a_2 can be made:

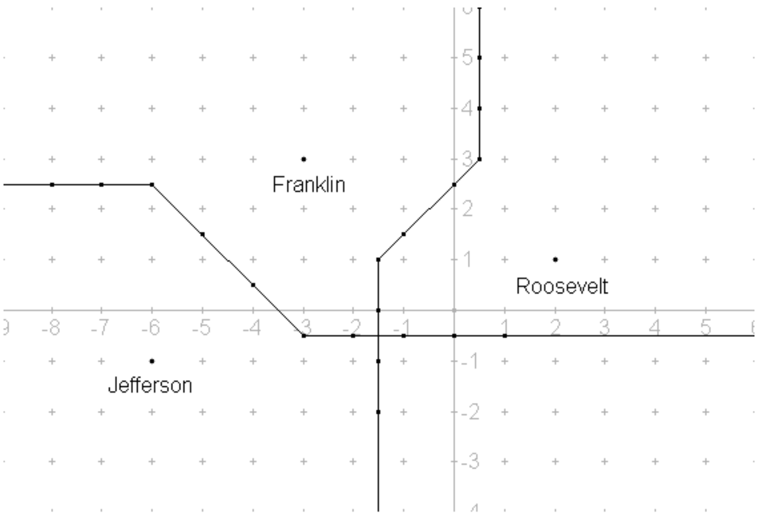
Because $x+3\geq 0$, when –3 ≤ x ≤ 2, you can replace $|x+3|$ by $x+3$.
Because $y-3>0$, when 3 < y, you can replace $|y-3|$ by $y-3$
Because $x-2\leq 0$, when –3 ≤ x ≤ 2, you can replace $|x-2|$ by $-x+2$
Because $y-1<0$, when 3 < y, you can replace $|y-1|$ by $y-1$

If these simplifications are put into the starting equation a_2 you can observe:

$$x+3+y-3=-x+2+y-1$$
$$x+y=-x+y+1$$
$$2x=1$$
$$x=1/2$$

x	y
½	3
½	4
½	5

The complete solution for section two was added to the solution of section 1 and can be seen below. The boundary was created such that $d(Franklin) = d(Roosevelt)$:



<u>In case IV – IX</u> of this section no solutions exist either. You don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points than the other.

<u>Section 3</u>

In section three you have a closer look at the boundary between Jefferson and Roosevelt school. Again you need the points where $d(Jefferson) = d(Roosevelt)$, meaning the points which are equal distance from both schools, to create the boundary. The equation a_3 to find these points is:

$$d(Jefferson) = d(Roosevelt)$$
$$d\left[(-6,-1),(x,y)\right] = d\left[(2,1),(x,y)\right]$$
$$|x+6| + |y+1| = |x-2| + |y-1|$$

To solve this equation easier for x and y we divide the calculation into different cases again:

	-1 ≤ y ≤ 1	-1 > y	1 < y
–6 ≤ x ≤ 2	Case I	Case II	Case III
2 < x	Case IV	Case V	Case VI

-6 > x	Case VII	Case IIX	Case IX

Case I: Under the terms –6 ≤ x ≤ 2 and -1 ≤ y ≤ 1 you can simplify the starting equation a_3 as following:

Because $x+6 \geq 0$, when –6 ≤ x ≤ 2, you can replace $|x+6|$ by $x+6$.

Because $y+1 \geq 0$, when -1 ≤ y ≤ 1, you can replace $|y+1|$ by $y+1$

Because $x-2 \leq 0$, when –6 ≤ x ≤ 2, you can replace $|x-2|$ by $-x+2$

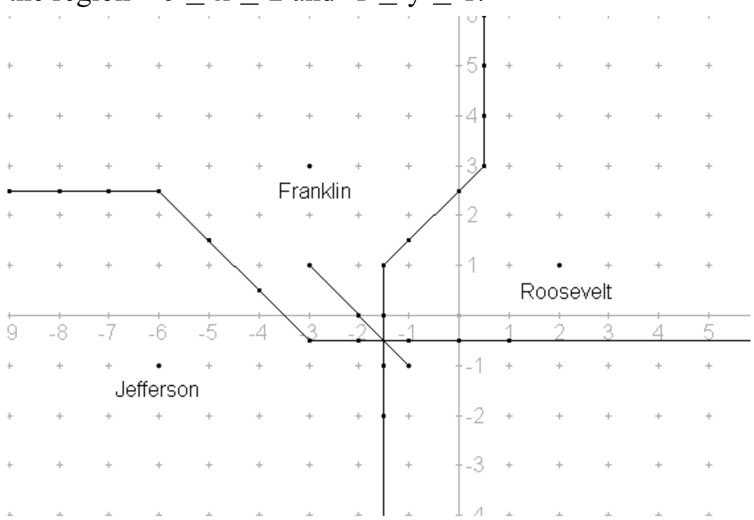
Because $y-1 \leq 0$, when -1 ≤ y ≤ 1, you can replace $|y-1|$ by $-y+1$

The simplified starting equation a_3 can be solved as following:

$$x+6+y+1=-x+2-y+1$$
$$x+y=-x-y-4$$
$$y=-x-2$$

x	y
-1	-1
-2	0
-3	1

In the picture below you can see the plotted boundary between Jefferson and Roosevelt school in the region –6 ≤ x ≤ 2 and -1 ≤ y ≤ 1:



Case II: Under these terms: –6 ≤ x ≤ 2 and -1 > y, you can simply a_3 as following:

Because $x+6 \geq 0$, when –6 ≤ x ≤ 2, you can replace $|x+6|$ by $x+6$.

Because $y+1 > 0$, when -1 > y, you can replace $|y+1|$ by $y+1$.

Because $x-2 \leq 0$, when –6 ≤ x ≤ 2, you can replace $|x-2|$ by $-x+2$.

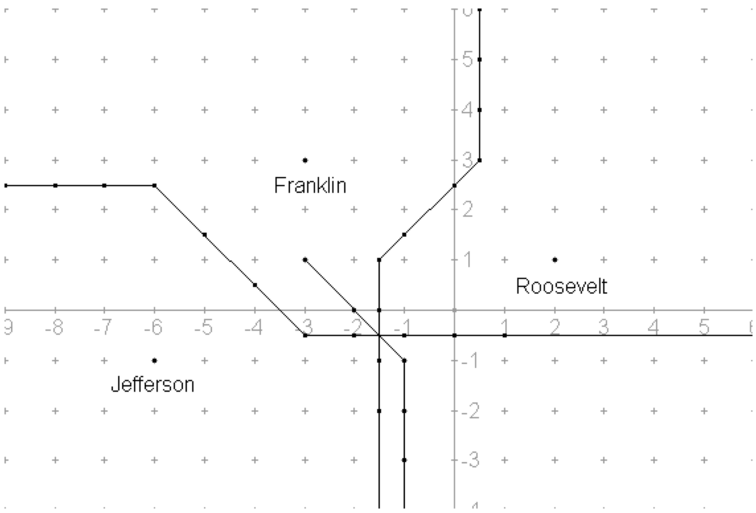
Because $y-1 < 0$, when -1 > y, you can replace $|y-1|$ by $-y+1$.

Substituted into a_3 you can observe:

$$x+6-y-1=-x+2-y+1$$
$$x-y+5=-x-y+3$$
$$2x=-2$$
$$x=-1$$

x	y
-1	-1
-1	-2
-1	-3
-1	...

Below you can see the plotted boundary:



Case III: If $-6 \le x \le 2$ and $1 < y$ you can simplify the equation a_3 from the beginning as following:

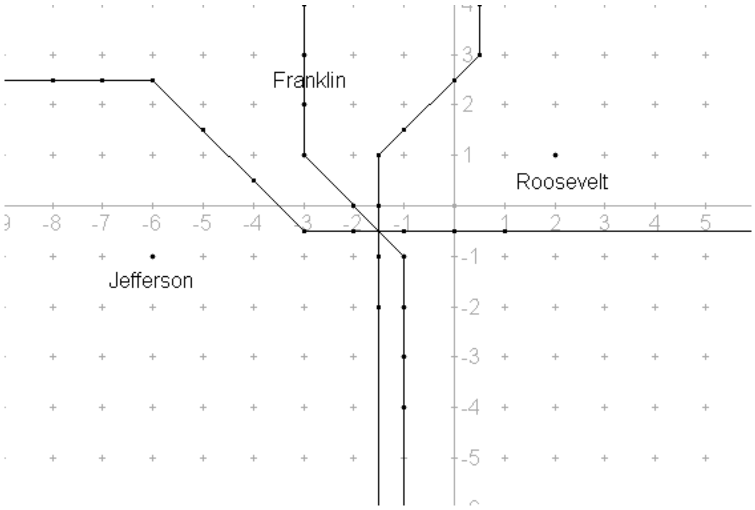
Because $x+6 \ge 0$, when $-6 \le x \le 2$, you can replace $|x+6|$ by $x+6$.
Because $y+1 > 0$, when $1 < y$, you can replace $|y+1|$ by $y+1$.
Because $x-2 \le 0$, when $-6 \le x \le 2$, you can replace $|x-2|$ by $-x+2$.
Because $y-1 < 0$, when $1 < y$, you can replace $|y-1|$ by $y-1$.

If these simplifications are substituted into a_3 you can solve the equation as following:

$$x+6+y+1=-x+2+y-1$$
$$x+y+7=-x+y+1$$
$$2x=-6$$
$$x=-3$$

x	y
-3	2
-3	3
-3	4
-3	...

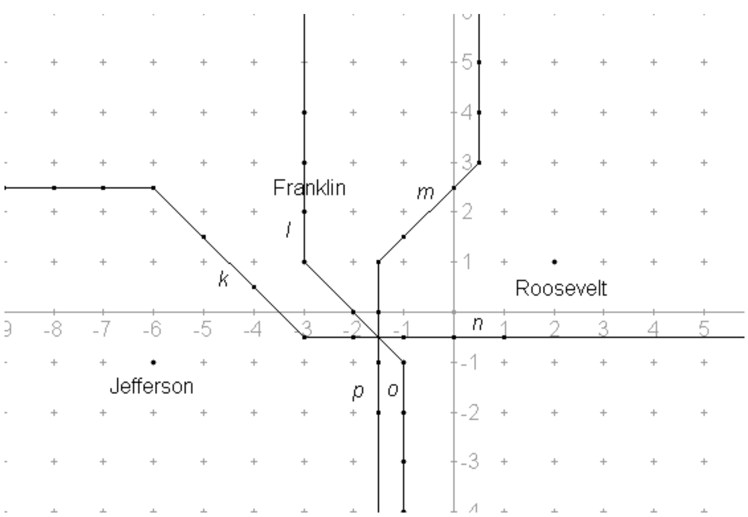
The last boundary which needed to be created was added to the other boundaries. The image which shows each calculated boundary looks like this:



<u>In case IV – IX</u> of this section no solutions exist either. Again you don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points than the other

<u>Section 4</u>

In section 4 we need to look at the information we got from the previous sections and use some basic logic to interpret it. In the last image above you can see all boundaries between the three schools. At first we look at the point where all boundaries intersect which is in this example (-1/2 / -1/2). From this point we will label every boundary for ease of explanation and look at each in particular:



At first we'll look at boundary *k.* Boundary *k* is a *correct* boundary because there is no other boundary which would divide Franklin and Jefferson school. On the contrary, boundary *l* was intended to divide Jefferson and Roosevelt but is a *wrong* boundary because the points on the line are closer to Franklin than to any other school. Since we now removed *l*, *m* must stay to maintain a boundary between Franklin and Roosevelt. Boundary *n* is not necessary and can be removed because it is part of the boundary which was created to divide Jefferson and Franklin. In addition, every points on this boundary are closest to Roosevelt. The same is true for boundary *p*. The points on *p* are closest to Jefferson and *p* is part of the boundary which was created to divide Franklin and Roosevelt, so *p* can be removed. To maintain a boundary between Jefferson and Roosevelt boundary *o* has to stay. After we have removed some of the boundaries we have now our final solution. The final solution to a school district such that every pupil attends the school they live closest to can be seen in the picture below: