# TAXICAB VERSIONS OF SOME EUCLIDEAN THEOREMS

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**Abstract**

In this paper, we give the taxicab versions of Pythagorean theorem, Stewart's theorem and a median property.

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**Key Words and Phrases:** Taxicab distance, Pythagorean theorem, Stewart's theorem, Median property

## 1 Introduction

The taxicab plane geometry introduced by Menger [7] and developed by Krause [6]. Now, there are about fifty articles published on the subject. The taxicab plane $\mathbb{R}_T^2$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. Taxicab distance between the points $P$ and $Q$ is the length of a shortest path from $P$ to $Q$ composed of the line segments parallel to the coordinate axes. That is, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then the taxicab distance from $P$ to $Q$ is $d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.

The taxicab plane geometry is non-Euclidean since it fails to satisfy the side-angle-side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry. Since the taxicab plane geometry has a different distance function it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors [1, 2, 3, 4, 5, 8, 9, 10, 11, 13]. The group of isometries that preserve taxicab distance is determined in [12].

Taxicab analogouses of Ceva's theorem, Menelaus' theorem and Thales' theorems are proven in [9]. Here in this study, we give taxicab versions of Stewart's theorem, Median property and Pythagorean theorem.

## 2 A Taxicab Version of the Stewart's Theorem

It is known that for any triangle $ABC$ in the Euclidean plane, if $X \in [BC]$ and $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, $p = d(B,X)$, $q = d(C,X)$, $x = d(A,X)$ then

$$x^2 = \frac{b^2 p + c^2 q}{p+q} - pq$$

which is known as **Stewart's theorem**. We use the following definitions given in [8] to give a taxicab version of this theorem.

Let $ABC$ be any triangle in the taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line $l$ is called a *base line* of $ABC$ if and only if
1. $l$ passes through a vertex,
2. $l$ is parallel to a coordinate axis,
3. $l$ intersects the opposite side (as a line segment) to the vertex in Condition1.

Clearly, at least one of the vertices of the triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex*. A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

The next theorem gives a taxicab version of the Stewart's theorem.

**Theorem 1.** *Let the sides of a triangle $ABC$ in the taxicab plane have lengths $\mathbf{a} = d_T(B,C)$, $\mathbf{b} = d_T(A,C)$ and $\mathbf{c} = d_T(A,B)$. If $X \in [BC]$ and $\mathbf{p} = d_T(B,X)$, $\mathbf{q} = d_T(C,X)$ and $\mathbf{x} = d_T(A,X)$, then*

$$\mathbf{x} = \begin{cases} \frac{\mathbf{bp}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has no base line through the vertex } A, \\ \frac{(\mathbf{b}-2\boldsymbol{\alpha})\mathbf{p}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has only one base line through the vertex } A, \text{ and } D \text{ is between } X \text{ and } C, \\ \frac{\mathbf{bp}+(\mathbf{c}-2\boldsymbol{\alpha})\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has only one base line through the vertex } A, \text{ and } D \text{ is between } X \text{ and } B, \\ \frac{(\mathbf{b}-2\boldsymbol{\alpha})\mathbf{p}+(\mathbf{c}-2\boldsymbol{\beta})\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has two base lines through the vertex } A, \text{ and } X \text{ is between the intersection points of the base lines and the opposite side, and } D \text{ is between } X \text{ and } C, \\ \frac{(\mathbf{b}-2\boldsymbol{\beta})\mathbf{p}+(\mathbf{c}-2\boldsymbol{\alpha})\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has two base lines through the vertex } A \text{ and } X \text{ is between the intersection points of the base lines and the opposite side, and } D \text{ is between } X \text{ and } B, \\ \frac{|\mathbf{bp}-\mathbf{cq}|}{\mathbf{p}+\mathbf{q}} & \text{If } ABC \text{ has two base lines through the vertex } A, \text{ and } X \text{ is not between the intersection points of the base lines and the opposite side.} \end{cases}$$

*where* $\boldsymbol{\alpha} = d_T(\textit{base line}, E)$, $\boldsymbol{\beta} = d_T(A, E')$ *and*

$D =$ *Intersection point of a base line and the opposite side,*
$E =$ *One of the vertices B and C such that D lies between X and E,*
$E' =$ *The point of orthogonal projection of the vertex distinct from A and E on the same base line.*

**Proof:** Let

$B' =$ Orthogonal projection of $B$ to the line through $A$ and parallel to $y$-axis,
$C' =$ Orthogonal projection of $C$ to the line through $A$ and parallel to $x$-axis,
$X' =$ Orthogonal projection of $X$ to the line through $A$ and parallel to $x$-axis,
$T =$ Orthogonal projection of $X$ to the line $CC'$,
$T' =$ Orthogonal projection of $X$ to the line $BB'$,

and $d(A, C') = b_1$ , $d(C, C') = b_2$ , $d(A, B') = c_2$ , $d(B, B') = c_1$.

Thus $\mathbf{b} = b_1 + b_2$ , $\mathbf{c} = c_1 + c_2$ and $\mathbf{x} = d(A, X') + d(X, X')$.

**Case I:** If $ABC$ is a triangle which has no base line through the vertex $A$ as in Fig.1, then one can easily obtain

$$\begin{aligned} \mathbf{q}\cdot(b_1 - c_1) &= (\mathbf{p} + \mathbf{q}) \cdot d(X, T) \\ \mathbf{p}\cdot(c_2 - b_2) &= (\mathbf{p} + \mathbf{q}) \cdot d(X, T') \end{aligned}$$

by [9, *Theo.*5]. Thus

$$\begin{aligned} d(X, T) &= \mathbf{q}\cdot(b_1 - c_1) / (\mathbf{p} + \mathbf{q}) \\ d(X, T') &= \mathbf{p}\cdot(c_2 - b_2) / (\mathbf{p} + \mathbf{q}) \end{aligned} .$$

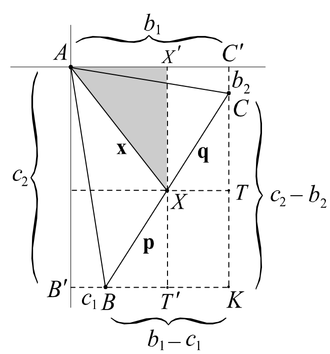


Figure 1

Since $d(A, X') = b_1 - d(X, T)$ , $d(X, X') = c_2 - d(X, T')$ and $\mathbf{x} = d(A, X') + d(X, X')$ we get

$$\begin{aligned}\mathbf{x} &= b_1 - \frac{\mathbf{q}(b_1 - c_1)}{\mathbf{p}+\mathbf{q}} + c_2 - \frac{\mathbf{p}(c_2 - b_2)}{\mathbf{p}+\mathbf{q}} = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} \quad .\end{aligned}$$

**Case II:** Let $ABC$ be a triangle which has only one base line through the vertex $A$. If $D$ is between $X$ and $C$ as in Fig.2, then one can easily obtain

$$\begin{aligned}\mathbf{q}\cdot(c_1 - b_1) &= (\mathbf{p}+\mathbf{q}) \cdot d(X,T) \\ \mathbf{p}\cdot(b_2 + c_2) &= (\mathbf{p}+\mathbf{q}) \cdot d(X,T')\end{aligned}$$

by [9, $Theo.$5]. Thus

$$\begin{aligned}d(X,T) &= \mathbf{q}\cdot(c_1 - b_1) \diagup (\mathbf{p}+\mathbf{q}) \\ d(X,T') &= \mathbf{p}\cdot(b_2 + c_2) \diagup (\mathbf{p}+\mathbf{q})\end{aligned} \quad .$$

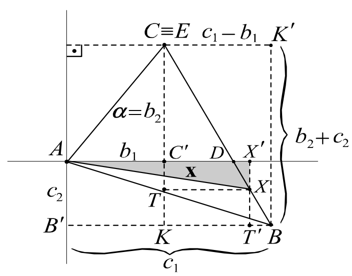


Figure 2

Since $d(A,X') = b_1 + d(X,T)$ , $d(X,X') = c_2 - d(X,T')$ and $\mathbf{x} = d(A,X') + d(X,X')$ we get

$$\begin{aligned}\mathbf{x} &= b_1 + \frac{\mathbf{q}(c_1 - b_1)}{\mathbf{p}+\mathbf{q}} + c_2 - \frac{\mathbf{p}(b_2 + c_2)}{\mathbf{p}+\mathbf{q}} = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2b_2\mathbf{p}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{(\mathbf{b} - 2\boldsymbol{\alpha})\mathbf{p}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} \quad .\end{aligned}$$

**Case III:** Let $ABC$ be a triangle which has only one base line through the vertex $A$. If $D$ is between $X$ and $B$ as in Fig.3, then one can easily obtain

$$\begin{aligned}\mathbf{q}\cdot(c_1 - b_1) &= (\mathbf{p}+\mathbf{q}) \cdot d(X,T) \\ \mathbf{p}\cdot(b_2 + c_2) &= (\mathbf{p}+\mathbf{q}) \cdot d(X,T')\end{aligned}$$

by [9, $Theo.$5]. Thus

$$\begin{aligned}d(X,T) &= \mathbf{q}\cdot(c_1 - b_1) \diagup (\mathbf{p}+\mathbf{q}) \\ d(X,T') &= \mathbf{p}\cdot(b_2 + c_2) \diagup (\mathbf{p}+\mathbf{q})\end{aligned} \quad .$$

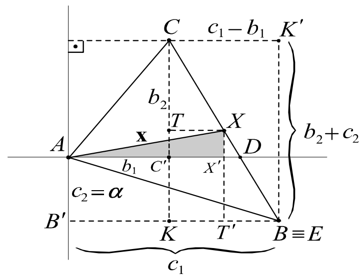


Figure 3

Since $d(A, X') = b_1 + d(X, T)$ , $d(X, X') = d(X, T') - c_2$ and $\mathbf{x} = d(A, X') + d(X, X')$ we get

$$\begin{aligned}\mathbf{x} &= b_1 + \frac{\mathbf{q}(c_1 - b_1)}{\mathbf{p} + \mathbf{q}} + \frac{\mathbf{p}(b_2 + c_2)}{\mathbf{p} + \mathbf{q}} - c_2 = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2c_2\mathbf{q}}{\mathbf{p} + \mathbf{q}} \\ &= \frac{\mathbf{b}\mathbf{p} + (\mathbf{c} - 2\boldsymbol{\alpha})\mathbf{q}}{\mathbf{p} + \mathbf{q}} \quad .\end{aligned}$$

**Case IV:** Let $ABC$ be a triangle which has two base lines through the vertex $A$, and $X$ be between the intersection points of the base lines and the opposite side. If $D$ is between $X$ and $C$ as in Fig.4, then one can easily obtain

$$\begin{aligned}\mathbf{q}\cdot(b_1 + c_1) &= (\mathbf{p} + \mathbf{q}) \cdot d(X, T) \\ \mathbf{p}\cdot(b_2 + c_2) &= (\mathbf{p} + \mathbf{q}) \cdot d(X, T')\end{aligned}$$

by [9, *Theo*.5]. Thus

$$\begin{aligned}d(X, T) &= \mathbf{q}\cdot(b_1 + c_1) / (\mathbf{p} + \mathbf{q}) \\ d(X, T') &= \mathbf{p}\cdot(b_2 + c_2) / (\mathbf{p} + \mathbf{q})\end{aligned} \quad .$$

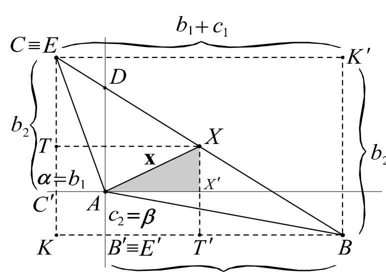


Figure 4

Since $d(A, X') = d(X, T) - b_1$ , $d(X, X') = d(X, T') - c_2$ and $\mathbf{x} = d(A, X') + d(X, X')$ we get

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{q}(b_1{+}c_1)}{\mathbf{p} + \mathbf{q}} - b_1 + \frac{\mathbf{p}(b_2{+}c_2)}{\mathbf{p} + \mathbf{q}} - c_2 = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2b_1\mathbf{p} - 2c_2\mathbf{q}}{\mathbf{p} + \mathbf{q}} \\ &= \frac{(\mathbf{b} - 2\boldsymbol{\alpha})\mathbf{p} + (\mathbf{c} - 2\boldsymbol{\beta})\mathbf{q}}{\mathbf{p} + \mathbf{q}} \quad .\end{aligned}$$

**Case V:** Let $ABC$ be a triangle which has two base lines through the vertex $A$, and $X$ be between the intersection points of the base lines and the opposite side. If $D$ is between $X$ and $B$ as in Fig.5, then one can easily obtain

$$\begin{array}{c} \mathbf{q}\cdot(b_1+c_1)=(\mathbf{p}+\mathbf{q})\cdot d(X,T) \\ \mathbf{p}\cdot(b_2+c_2)=(\mathbf{p}+\mathbf{q})\cdot d(X,T') \end{array}$$

by [9, $Theo.5$]. Thus

$$\begin{array}{c} d(X,T)=\mathbf{q}\cdot(b_1+c_1)/(\mathbf{p}+\mathbf{q}) \\ d(X,T')=\mathbf{p}\cdot(b_2+c_2)/(\mathbf{p}+\mathbf{q}) \end{array} .$$

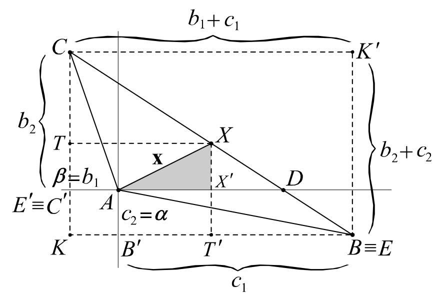


Figure 5

Since $d(A,X')=d(X,T)-b_1$ , $d(X,X')=d(X,T')-c_2$ and
$\mathbf{x}=d(A,X')+d(X,X')$ we get

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}}-b_1+\frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}}-c_2=\frac{b_1\mathbf{p}+b_2\mathbf{p}+c_1\mathbf{q}+c_2\mathbf{q}-2b_1\mathbf{p}-2c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{(\mathbf{b}-2\boldsymbol{\beta})\mathbf{p}+(\mathbf{c}-2\boldsymbol{\alpha})\mathbf{q}}{\mathbf{p}+\mathbf{q}} \quad . \end{aligned}$$

**Case VI:** Let $ABC$ be a triangle which has two base lines passing through the vertex $A$. If $X$ is not between the intersection points of the base lines and the opposite side as in Fig.6 and Fig.7, then one can easily obtain

$$\begin{array}{c} \mathbf{q}\cdot(b_1+c_1)=(\mathbf{p}+\mathbf{q})\cdot d(X,T) \\ \mathbf{p}\cdot(b_2+c_2)=(\mathbf{p}+\mathbf{q})\cdot d(X,T') \end{array}$$

by [9, $Theo.5$]. Thus

$$\begin{array}{c} d(X,T)=\mathbf{q}\cdot(b_1+c_1)/(\mathbf{p}+\mathbf{q}) \\ d(X,T')=\mathbf{p}\cdot(b_2+c_2)/(\mathbf{p}+\mathbf{q}) \end{array} .$$

Now, two subcases are possible. If $D$ is between $X$ and $C$ as in Fig.6, then $d(A,X')=d(X,T)-b_1$ , $d(X,X')=c_2-d(X,T')$ and we get

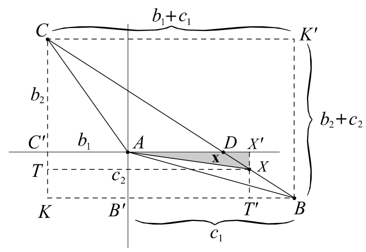


Figure 6

$$\begin{aligned}\mathbf{x} &= d(A,X') + d(X,X') = \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}} - b_1 + c_2 - \frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}} \\ &= \frac{c_1\mathbf{q} + c_2\mathbf{q} - b_1\mathbf{p} - b_2\mathbf{p}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{cq}-\mathbf{bp}}{\mathbf{p}+\mathbf{q}} \quad .\end{aligned}$$

If $D$ is between $X$ and $B$ as in Fig.7, then $d(A,X') = b_1 - d(X,T)$ , $d(X,X') = d(X,T') - c_2$ and we get

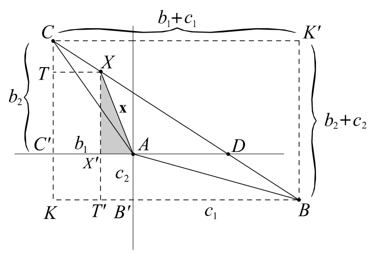


Figure 7

$$\begin{aligned}\mathbf{x} &= d(A,X') + d(X,X') = b_1 - \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}} + \frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}} - c_2 \\ &= \frac{b_1\mathbf{p} + b_2\mathbf{p} - c_1\mathbf{q} - c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{bp}-\mathbf{cq}}{\mathbf{p}+\mathbf{q}} \quad .\end{aligned}$$

Consequently, $\mathbf{x} = \dfrac{|\mathbf{bp}-\mathbf{cq}|}{\mathbf{p}+\mathbf{q}}$ which completes the proof.

If $X$ is the midpoint of $[BC]$ of any triangle $ABC$ in the Euclidean plane with $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$ and $V_a = d(A,X)$ then

$$2V_a^2 = b^2 + c^2 - a^2 / 2$$

which is known as **Median property**. The following corollary gives a taxicab version of this property, for $\mathbf{p} = \mathbf{q}$ in Theorem 1 :

**Corollary 2.** *Let the sides of a triangle $ABC$ in the taxicab plane have lengths $\mathbf{a} = d_T(B,C)$, $\mathbf{b} = d_T(A,C)$ and $\mathbf{c} = d_T(A,B)$. If $X$ is the midpoint* *of* $[BC]$ and $V_{\mathbf{a}} = d_T(A, X)$, *then*

$$2V_{\mathbf{a}} = \begin{cases} \mathbf{b}+\mathbf{c} & \text{If } ABC \text{ has no base line through the vertex } A, \\ \mathbf{b}+\mathbf{c} - 2\boldsymbol{\alpha} & \text{If } ABC \text{ has only one base line through the vertex } A, \\ \mathbf{b}+\mathbf{c} - 2(\boldsymbol{\alpha}+\boldsymbol{\beta}) & \text{If } ABC \text{ has two base lines through the vertex } A \\ & \text{and } X \text{ is between the intersection points of the base} \\ & \text{lines and the opposite side,} \\ |\mathbf{b} - \mathbf{c}| & \text{If } ABC \text{ has two base lines through the vertex } A \\ & \text{and } X \text{ is not between the intersection points of the} \\ & \text{base lines and the opposite side,} \end{cases}$$

*where* $\boldsymbol{\alpha} = d_T(\textit{base line}, E)$, $\boldsymbol{\beta} = d_T(A, E')$ *and*

$D =$ *Intersection point of a base line and the opposite side,*
$E =$ *One of the vertices B and C such that D lies between X and E,*
$E' =$ *The point of orthogonal projection of the vertex distinct from A and E on the same base line.*

# 3 A Taxicab Version of the Pythagorean Theorem

It is well known that for any right triangle $ABC$ in the Euclidean plane, if $[BC]$ is its hypotenuse and $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$ then

$$a^2 = b^2 + c^2$$

which is known as **Pythagorean theorem**. A taxicab version of this theorem can be stated as follows:

**Theorem 3.** *Let* $\mathbf{a}$ *denote the length of the hypotenuse,* $\mathbf{b}$ *and* $\mathbf{c}$ *denote the lengths of the legs of a triangle* $ABC$ *with right angle* $A$ *in the taxicab plane. Then*

$$\mathbf{a} = \begin{cases} \mathbf{b} + \mathbf{c} - 2\boldsymbol{\gamma} & \text{If there exists only one base line through the vertex } A, \\ \mathbf{b} + \mathbf{c} & \text{If there exist two base lines through the vertex } A, \end{cases}$$

*where* $\boldsymbol{\gamma} = d_T(A, H)$ *and*
$H =$ *The point of orthogonal projection of B or C to the base segment through A.*

**Proof:** $A$ is a basic vertex since $ABC$ is a triangle with right angle $A$. That is, this triangle always has one or two base lines passing through $A$.
**Case I:** Let $b_1$, $b_2$, $c_1$ and $c_2$ denote the parameters used in the proof of Theo.1. If there exists only one base line through the vertex $A$ as in Fig.8, and

$\mathbf{b} = d(A, C') + d(C, C') = b_1 + b_2$, $\mathbf{c} = d(A, B') + d(B, B') = c_2 + c_1$ then

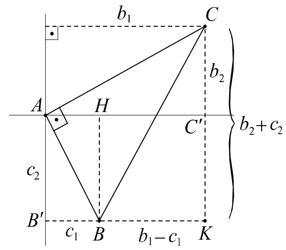


Figure 8

$$\begin{aligned} \mathbf{a} &= b_1 - c_1 + b_2 + c_2 = \mathbf{b} + \mathbf{c} - 2c_1 \quad , \quad c_1 = \boldsymbol{\gamma} = d(B, B') = d(A, H) \\ &= \mathbf{b} + \mathbf{c} - 2\boldsymbol{\gamma} \ . \end{aligned}$$

**Case II:** If there exist two base lines through the vertex $A$, then the basic lines coincide with the perpendicular sides of $ABC$ as in Fig.9.Thus, obviously, $\mathbf{a} = \mathbf{b} + \mathbf{c}$ which completes the proof.

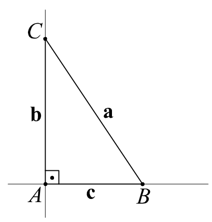


Figure 9

# 4 Another Taxicab Version of the Pythagorean Theorem

A taxicab version of the Pythagorean theorem has been given in section 3 using a parameter $\boldsymbol{\gamma}$ which is length of a part of the base segment. In this section, we use slopes of hypotenuse and a side of the right triangle to give another version of the theorem.

**Theorem 4.** *Let* $\mathbf{a}$ *denote the length of the hypotenuse,* $\mathbf{b}$ *and* $\mathbf{c}$ *denote the lengths of the legs of a right triangle in the taxicab plane. If the slope of the hypotenuse is* $m_1$ *and the slope of the anyone of the legs is* $m_2$*, then*

$$\mathbf{a}^2 = \rho(m_1, m_2) \cdot \left(\mathbf{b}^2 + \mathbf{c}^2\right)$$

*where*

$$\rho(m_1, m_2) = \begin{cases} \left(\frac{1+m_2^2}{1+m_1^2}\right)\left(\frac{1+|m_1|}{1+|m_2|}\right)^2 & , if \;\; m_1, m_2 \in \mathbb{R} \\ \left(\frac{1+m_2^2}{(1+|m_2|)^2}\right) & , if \;\; m_1 \to \infty \\ \left(\frac{(1+|m_1|)^2}{1+m_1^2}\right) & , if \;\; m_2 \to \infty \end{cases}$$

**Proof:** We know from [4] that for any two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the taxicab plane, if $x_1 \neq x_2$ then

$$d(P,Q) = \left[(1+m^2)^{1/2} \big/ (1+|m|)\right] \cdot d_T(P,Q)$$

where $m = (y_1 - y_2)\big/(x_1 - x_2)$ and if $x_1 = x_2$, that is, $m \to \infty$ then

$$d(P,Q) = d_T(P,Q)$$

which allows us to convert a taxicab distance to the Euclidean distance.

Let $a$, $b$ and $c$ be the corresponding Euclidean lengths of the sides of the same right triangle, $m_1$ denote the slope of the hypotenuse, and $m_2$ denote the slope of the anyone of the legs. If $m_2 \neq 0$, then the slope of the other leg is $(-1\big/m_2)$ and

$$a = \left[(1+m_1^2)^{1/2} \big/ (1+|m_1|)\right] \cdot \mathbf{a}$$

$$b = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{b}$$

$$c = \left[\left(1+\left(-\tfrac{1}{m_2}\right)^2\right)^{1/2} \Big/ \left(1+\left|-\tfrac{1}{m_2}\right|\right)\right] \cdot \mathbf{c} = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{c}$$

If $m_2 = 0$, then the slope of the other leg is $(-1\big/m_2) \to \infty$ or if $m_2 \to \infty$, then the slope of the other leg is $(-1\big/m_2) \to 0$ and

$$a = \left[(1+m_1^2)^{1/2} \big/ (1+|m_1|)\right] \cdot \mathbf{a}$$

$$b = \mathbf{b}$$

$$c = \mathbf{c} \quad .$$

If $m_1 \to \infty$, then

$$a = \mathbf{a}$$

$$b = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{b}$$

$$c = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{c} \quad .$$

Using these values of $a$, $b$ and $c$ in the Euclidean Pythagorean theorem one obtains

$$\mathbf{a}^2 = \rho(m_1, m_2) \cdot \left(\mathbf{b}^2 + \mathbf{c}^2\right)$$

where

$$\rho(m_1, m_2) = \begin{cases} \left(\frac{1+m_2^2}{1+m_1^2}\right)\left(\frac{1+|m_1|}{1+|m_2|}\right)^2 & , \textit{if} \quad m_1, m_2 \in \mathbb{R} \\ \left(\frac{1+m_2^2}{(1+|m_2|)^2}\right) & , \textit{if} \quad m_1 \to \infty \\ \left(\frac{(1+|m_1|)^2}{1+m_1^2}\right) & , \textit{if} \quad m_2 \to \infty \end{cases}$$

which completes the proof.

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