ON THE RATIO OF DIRECTED LENGTHS IN THE TAXICAB PLANE AND RELATED PROPERTIES

Münevver Özcan and Rüstem Kaya

Abstract. In this work, it is shown that a point of division divides a related line segment in the same ratio both in the taxicab and Euclidean planes. Consequently, the coordinates of the division point can be determined by the same formula as in the Euclidean plane. In the latter parts of the paper, taxicab analogues of Ceva's and Menelaus' Theorems and the theorem of directed lines are given.

1. Introduction. A family of "metrics", including the taxicab metric, have been published by H. Minkowski [9] at the beginning of the last century. Later, taxicab plane geometry was introduced in [8] and developed in [5] using the *taxicab* metric in the coordinate plane by

$$d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

instead of the Euclidean metric

$$d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.

A few problems related to the taxicab geometry have been studied and improved by some authors, see [1, 2, 3, 4, 7, 10, 11, 12, 13, 14]. The taxicab geometry was constructed by simply replacing the Euclidean distance function d_E by the taxicab distance function d_T. Therefore it seems interesting to study the taxicab analogues of the topics which include the concept of distance in Euclidean geometry. These topics are division point, directed lengths, ratio of directed lengths, Menelaus' Theorem, Ceva's Theorem, and the theorem of directed lines.

2. Directed Taxicab Length and Division Point. Let X and Y be any two points on a directed straight line l. We define *directed taxicab length* of the line segment XY as follows:

$$d_T[XY] = \begin{cases} d_T(X,Y), & \text{if } XY \text{ and } l \text{ have the same direction} \\ -d_T(X,Y), & \text{if } XY \text{ and } l \text{ have opposite direction.} \end{cases}$$

Thus, $d_T[XY] = -d_T[YX]$. Clearly, *directed length* in the Euclidean plane can be defined in a similar way. That is

$$d_E[XY] = \begin{cases} d_E(X,Y), & \text{if } XY \text{ and } l \text{ have the same direction} \\ -d_E(X,Y), & \text{if } XY \text{ and } l \text{ have opposite direction.} \end{cases}$$

If A, B, C are points on a same directed line and C is between points A and B, we denote this by ACB. If ACB, then C divides the line segment AB *internally* and the *ratio* of the directed taxicab lengths is a positive real number, that is $d_T[AC]/d_T[CB] = \lambda > 0$. If ABC or CAB then C divides the line segment AB *externally*, and $d_T[AC]/d_T[CB] = \lambda < 0$, that is, the line segments AC and CB have opposite directions. In both cases C is called the *division point* which divides the line segment AB in ratio λ.
Clearly, $C \neq B$. $C = A \Leftrightarrow \lambda = 0$ and (C is at infinity $\Leftrightarrow \lambda = -1$).

Let C and C' be two points such that C divides a given line segment AB internally and C' divides AB externally in the *same proportion* though with opposite signs. Thus, the ratio of the directed lengths, $d_T[AC]/d_T[CB] = -d_T[AC']/d_T[C'B]$ is the same positive number λ.

Theorem 1. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be any two distinct points in the analytical plane. If $Q = (x, y)$ is a point on the line passing through P_1 and P_2, then

$$d_T[P_1Q]/d_T[QP_2] = d_E[P_1Q]/d_E[QP_2].$$

That is, the ratios of the Euclidean and taxicab directed lengths are the same.

Proof. If $Q = P_1$ then both ratios are equal to 0. If Q is at infinity then both ratios are equal to -1. Therefore without loss of generality, let $P_1 \neq Q \neq P_2$. It is enough to show that

$$\frac{|x_1 - x| + |y_1 - y|}{|x - x_2| + |y - y_2|} = \frac{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}}{\sqrt{(x - x_2)^2 + (y - y_2)^2}} \quad . \tag{1}$$

Squaring both sides of Equation (1) one obtains

$$\frac{|x_1 - x|^2 + |y_1 - y|^2 + 2|x_1 - x||y_1 - y|}{|x - x_2|^2 + |y - y_2|^2 + 2|x - x_2||y - y_2|} = \frac{(x_1 - x)^2 + (y_1 - y)^2}{(x - x_2)^2 + (y - y_2)^2}$$

which is equivalent to

$$\frac{\left[(x-x_2)^2+(y-y_2)^2\right]\left[|x_1-x|^2+|y_1-y|^2+2\,|x_1-x|\,|y_1-y|\right]}{\left[(x_1-x)^2+(y_1-y)^2\right]\left[|x-x_2|^2+|y-y_2|^2+2\,|x-x_2|\,|y-y_2|\right]}=1.$$

Rearranging the last equality one gets

$$\frac{\left[(x_1-x)^2+(y_1-y)^2\right]\left[(x-x_2)^2+(y-y_2)^2\right]+2|x_1-x||y_1-y|\left[(x-x_2)^2+(y-y_2)^2\right]}{\left[(x_1-x)^2+(y_1-y)^2\right]\left[(x-x_2)^2+(y-y_2)^2\right]+2|x-x_2||y-y_2|\left[(x_1-x)^2+(y_1-y)^2\right]}=1$$

which means that

$$\frac{2\,|x_1-x|\,|y_1-y|\left[(x-x_2)^2+(y-y_2)^2\right]}{2\,|x-x_2|\,|y-y_2|\left[(x_1-x)^2+(y_1-y)^2\right]}=1$$

or simply

$$\frac{|x_1-x|\,|y_1-y|}{|x-x_2|\,|y-y_2|}=\frac{(x_1-x)^2+(y_1-y)^2}{(x-x_2)^2+(y-y_2)^2}\ . \tag{2}$$

Examining the left side of Equation (2) one obtains

$$\frac{|x_1-x|\,|y_1-y|}{|x-x_2|\,|y-y_2|}=\frac{(x_1-x)\,(y_1-y)}{(x-x_2)\,(y-y_2)} \tag{3}$$

for all positions of Q on P_1P_2. Using Equation (3) in Equation (2) one obtains

$$(x_1-x)(y_1-y)\left[(x-x_2)^2+(y-y_2)^2\right]$$
$$=(x-x_2)(y-y_2)\left[(x_1-x)^2+(y_1-y)^2\right]$$

which can be expressed as follows:

$$
\begin{aligned}
&(x_1 - x)(x - x_2)\left[(x - x_2)(y_1 - y) + (x_1 - x)(y - y_2)\right] \\
&\qquad = (y_1 - y)(y - y_2)\left[(x - x_2)(y_1 - y) + (x_1 - x)(y - y_2)\right].
\end{aligned}
$$

Rearranging this equality one gets

$$
\left[(x - x_2)(y_1 - y) - (x_1 - x)(y - y_2)\right]\left[(x_1 - x)(x - x_2) - (y_1 - y)(y - y_2)\right] = 0 \tag{4}
$$

If $x_1 = x_2$ then $x = x_1 = x_2$ and Equation (4) is obvious. If $x_1 \neq x_2$ then

$$
y = \left[(x_2 - x)y_1 - (x_1 - x)y_2\right] / \; (x_2 - x_1)
$$

since Q is on the line P_1P_2. Now, using this value of y in the first bracket of Equation (4) we get

$$
\begin{aligned}
&(x - x_2)(y_1 - y) - (x_1 - x)(y - y_2) \\
&= (x - x_2)\left(y_1 - \tfrac{(x_2 - x)y_1 - (x_1 - x)y_2}{x_2 - x_1}\right) - (x_1 - x)\left(\tfrac{(x_2 - x)y_1 - (x_1 - x)y_2}{x_2 - x_1} - y_2\right) \\
&= \tfrac{1}{x_2 - x_1}\left[(x - x_2)(xy_1 - xy_2 + x_1y_2 - x_1y_1) - (x_1 - x)(xy_2 - xy_1 + x_2y_1 - x_2y_2)\right] \\
&= \tfrac{1}{x_2 - x_1}\left[(x - x_1)(x - x_2)(y_1 - y_2) - (x_1 - x)(x - x_2)(y_2 - y_1)\right] = 0
\end{aligned}
$$

which shows that Equation (4) is satisfied.

The following corollary shows how one can find the coordinates of the division point which divides the line segment joining two given points in a given ratio, in the taxicab plane.

Corollary. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two distinct points in the taxicab plane. If $Q = (x, y)$ divides the line segment P_1P_2 in ratio λ then,

$$
x = \frac{x_1 + \lambda x_2}{1 + \lambda} \quad , \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda} \; ; \qquad \lambda \in \mathbb{R}, \; \lambda \neq -1
$$

as in the Euclidean plane.

Proof. Although the Corollary follows from Theorem 1 we prefer to give a direct proof. The given formula is obvious when $\lambda = 0$ or $\lambda = -1$. If $\lambda \neq 0, -1$ and Q divides the line segment P_1P_2 in ratio λ, we have $\left| d_T[P_1Q_1] / d_T[Q_1P_2] \right| = |\lambda|$. That is,

$$\frac{|x_1 - x| + |y_1 - y|}{|x - x_2| + |y - y_2|} = |\lambda| \ . \tag{5}$$

Since $P_1 \neq P_2$,

$$|\lambda| = |\lambda| \left(\frac{|x_1 - x_2| + |y_1 - y_2|}{|x_1 - x_2| + |y_1 - y_2|} \right) = \frac{|\lambda x_1 - \lambda x_2| + |\lambda y_1 - \lambda y_2|}{|x_1 - x_2| + |y_1 - y_2|} \ .$$

Adding $x_1 - x_1$ and $y_1 - y_1$ to the first and second summands in the numerator and similarly $\lambda x_2 - \lambda x_2$ and $\lambda y_2 - \lambda y_2$ in the denominator respectively, one obtains

$$|\lambda| = \frac{|\lambda x_1 + x_1 - x_1 - \lambda x_2| + |\lambda y_1 + y_1 - y_1 - \lambda y_2|}{|x_1 + \lambda x_2 - \lambda x_2 - x_2| + |y_1 + \lambda y_2 - \lambda y_2 - y_2|} \ .$$

Multiplying the numerator and the denominator of the right side of the last statement by $1/|1+\lambda|$, one gets

$$\begin{aligned}
|\lambda| &= \frac{\left| \dfrac{\lambda x_1 + x_1 - x_1 - \lambda x_2}{1+\lambda} \right| + \left| \dfrac{\lambda y_1 + y_1 - y_1 - \lambda y_2}{1+\lambda} \right|}{\left| \dfrac{x_1 + \lambda x_2 - \lambda x_2 - x_2}{1+\lambda} \right| + \left| \dfrac{y_1 + \lambda y_2 - \lambda y_2 - y_2}{1+\lambda} \right|} \\
&= \frac{\left| \dfrac{(1+\lambda)x_1}{1+\lambda} - \dfrac{x_1 + \lambda x_2}{1+\lambda} \right| + \left| \dfrac{(1+\lambda)y_1}{1+\lambda} - \dfrac{y_1 + \lambda y_2}{1+\lambda} \right|}{\left| \dfrac{x_1 + \lambda x_2}{1+\lambda} - \dfrac{(1+\lambda)x_2}{1+\lambda} \right| + \left| \dfrac{y_1 + \lambda y_2}{1+\lambda} - \dfrac{(1+\lambda)y_2}{1+\lambda} \right|} \\
&= \frac{\left| x_{1-} \dfrac{x_1 + \lambda x_2}{1+\lambda} \right| + \left| y_{1-} \dfrac{y_1 + \lambda y_2}{1+\lambda} \right|}{\left| \dfrac{x_1 + \lambda x_2}{1+\lambda} - x_2 \right| + \left| \dfrac{y_1 + \lambda y_2}{1+\lambda} - y_2 \right|} \ .
\end{aligned}$$

Comparing this result with Equation (5) we obtain

$$x = \frac{x_1 + \lambda x_2}{1+\lambda} \quad \text{and} \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}.$$

3. Theorems of Menelaus and Ceva in the Taxicab Plane. In this section, the taxicab analogues of the Theorems of Menelaus and Ceva are studied. In fact, the validity of these theorems is clear from the Theorem 1, but we prefer to state and give partial proofs for them.

Theorem 2. (*Menelaus' Theorem.*) Let $\{P_1, P_2, P_3\}$ be a triangle and Q_1, Q_2, Q_3 be on the lines that contain the sides P_1P_2, P_2P_3, P_3P_1 respectively, in the taxicab plane. If Q_1, Q_2, Q_3 are collinear, then

$$\frac{d_T\,[P_1Q_1]}{d_T\,[Q_1P_2]} \cdot \frac{d_T\,[P_2Q_2]}{d_T\,[Q_2P_3]} \cdot \frac{d_T\,[P_3Q_3]}{d_T\,[Q_3P_1]} = -1 \tag{6}$$

where none of Q_1, Q_2, Q_3 coincide with any of P_1, P_2, P_3.

Proof. Several cases are possible, according to the positions of points P_1, P_2, P_3 and Q_1, Q_2, Q_3. We give a proof of the theorem only in the following special case.

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$ and $x_i \neq x_{i+1}$ and let Q_1, Q_2, Q_3 be on a line l given by $y = mx + k$ such that $Q_i = l \wedge P_iP_{i+1 \pmod 3}$ and l is not parallel to the line P_iP_{i+1}, for $i = 1, 2, 3$ (Figure 1) . Clearly $mx_i - y_i + k \neq 0$ since $P_i \neq Q_j$ for i, $j = 1, 2, 3$ and $m \neq (y_{i+1} - y_i)(x_{i+1} - x_i)^{-1}$. The equation of the line P_iP_{i+1} is given by

$$y = (y_{i+1} - y_i)(x_{i+1} - x_i)^{-1}x - (x_iy_{i+1} - x_{i+1}y_i)(x_{i+1} - x_i)^{-1}.$$

It follows from a simple calculation that

$$Q_i = \left(\frac{x_iy_{i+1} - x_{i+1}y_i - kx_i + kx_{i+1}}{mx_i - mx_{i+1} - y_i + y_{i+1}}, \frac{mx_iy_{i+1} - mx_{i+1}y_i - ky_i + ky_{i+1}}{mx_i - mx_{i+1} - y_i + y_{i+1}} \right).$$

Now let us find $\dfrac{d_T\,[P_iQ_i]}{d_T\,[Q_iP_{i+1}]}$.

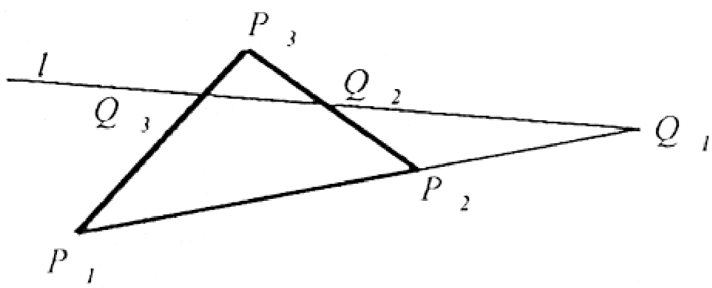

Figure 1.

$$\frac{d_T\left[P_1Q_1\right]}{d_T\left[Q_1P_2\right]} = -\frac{d_T\left(P_1,Q_1\right)}{d_T\left(Q_1,P_2\right)}$$

$$= -\frac{\left|x_1 - \frac{x_1y_2-x_2y_1-kx_1+kx_2}{mx_1-mx_2-y_1+y_2}\right| + \left|y_1 - \frac{mx_1y_2-mx_2y_1-ky_1+ky_2}{mx_1-mx_2-y_1+y_2}\right|}{\left|\frac{x_1y_2-x_2y_1-kx_1+kx_2}{mx_1-mx_2-y_1+y_2} - x_2\right| + \left|\frac{mx_1y_2-mx_2y_1-ky_1+ky_2}{mx_1-mx_2-y_1+y_2} - y_2\right|}$$

$$= -\frac{\left|mx_1^2-mx_1x_2-x_1y_1+x_2y_1-kx_2+kx_1\right|+\left|mx_1y_1-mx_1y_2+y_1y_2-y_1^2-ky_2+ky_1\right|}{\left|mx_2^2-mx_1x_2-x_2y_2+x_1y_2-kx_1+kx_2\right|+\left|mx_2y_2-mx_2y_1+y_1y_2-y_2^2-ky_1+ky_2\right|}$$

$$= -\frac{|x_1(mx_1-y_1+k)-x_2(mx_1-y_1+k)|+|y_1(mx_1-y_1+k)-y_2(mx_1-y_1+k)|}{|x_2(mx_2-y_2+k)-x_1(mx_2-y_2+k)|+|y_2(mx_2-y_2+k)-y_1(mx_2-y_2+k)|}$$

$$= -\frac{\left(|x_1-x_2|+|y_1-y_2|\right)|mx_1-y_1+k|}{\left(|x_1-x_2|+|y_1-y_2|\right)|mx_2-y_2+k|}$$

$$= -\frac{|mx_1-y_1+k|}{|mx_2-y_2+k|}.$$

Similarly,

$$\frac{d_T\left[P_2Q_2\right]}{d_T\left[Q_2P_3\right]} = \frac{d_T(P_2,Q_2)}{d_T(Q_2,P_3)} = \frac{|mx_2-y_2+k|}{|mx_3-y_3+k|}$$

and

$$\frac{d_T\,[P_3Q_3]}{d_T\,[Q_3P_1]} = \frac{d_T(P_3, Q_3)}{d_T(Q_3, P_1)} = \frac{|mx_3 - y_3 + k|}{|mx_1 - y_1 + k|}$$

and consequently,

$$\frac{d_T\,[P_iQ_i]}{d_T\,[Q_iP_{i+1}]} = s\frac{|mx_i - y_i + k|}{|mx_{i+1} - y_{i+1} + k|}\,, \quad s = \begin{cases} -1, & \text{if } i = 1 \\ 1, & \text{if } i = 2, 3. \end{cases}$$

Now, it can be easily computed that

$$\prod_{i=1}^{3} (d_T\,[P_iQ_i]\,/d_T\,[Q_iP_{i+1}]) = -1.$$

Theorem 3. (*Converse of Menelaus' Theorem.*) Let $\{P_1, P_2, P_3\}$ be a triangle and Q_1, Q_2, Q_3 be three points on the lines that contain the sides $P_1P_2,\ P_2P_3,\ P_3P_1$, respectively, in the taxicab plane. If

$$\frac{d_T\,[P_1Q_1]}{d_T\,[Q_1P_2]} \cdot \frac{d_T\,[P_2Q_2]}{d_T\,[Q_2P_3]} \cdot \frac{d_T\,[P_3Q_3]}{d_T\,[Q_3P_1]} = -1,$$

then Q_1, Q_2, Q_3 are collinear. Note that none of Q_1, Q_2, Q_3 are P_1, P_2, P_3.

Theorem 4. (*Ceva's Theorem.*) Let $\{P_1, P_2, P_3\}$ be a triangle and lines l_1, l_2, l_3 pass through the vertices $P_1,\ P_2,\ P_3$, respectively and intersect lines containing the opposite sides at points Q_1, Q_2, Q_3. The lines l_1, l_2, l_3 are concurrent (or parallel) if and only if

$$\frac{d_T\,[P_1Q_3]}{d_T\,[Q_3P_2]} \cdot \frac{d_T\,[P_2Q_1]}{d_T\,[Q_1P_3]} \cdot \frac{d_T\,[P_3Q_2]}{d_T\,[Q_2P_1]} = 1.$$

Note that none of Q_1, Q_2, Q_3 are P_1, P_2, P_3.

4. Theorems of Directed Lines (Strahlensätze). In general, it is well–known that the axiom of congruence and consequently properties of similarity for triangles are not valid in the taxicab plane. But, it follows from Theorem 1 that the following *directed line theorem* [6] is valid in it.

Theorem 5. Let a pencil of lines be intersected by a family of parallel lines in the taxicab plane (see Figure 2).

(i) The ratios of the directed lengths of the corresponding segments on the lines belonging to the pencil are the same. For example,

$$\begin{aligned} d_T\,[SA] : d_T\,[SB] : d_T\,[SC] &= d_T\,[SA_1] : d_T\,[SB_1] : d_T\,[SC_1] \\ &= d_T\,[SA_2] : d_T\,[SB_2] : d_T\,[SC_2] \end{aligned}$$

or

$$d_T\,[SA_1] : d_T\,[SB_1] = d_T\,[A_1A_2] : d_T\,[B_1B_2]\;.$$

(ii) The ratios of the directed lengths of line segments on the parallel lines and corresponding segments on the lines belonging to the pencil, which are measured from the vertex, are the same. For example,

$$\begin{aligned} d_T\,[CB] : d_T\,[C_1B_1] : d_T\,[C_2B_2] &= d_T\,[SC] : d_T\,[SC_1] : d_T\,[SC_2] \\ &= d_T\,[SB] : d_T\,[SB_1] : d_T\,[SB_2] \end{aligned}$$

or

$$\begin{aligned} d_T\,[AB] : d_T\,[A_1B_1] : d_T\,[A_2B_2] &= d_T\,[SA] : d_T\,[SA_1] : d_T\,[SA_2] \\ &= d_T\,[SB] : d_T\,[SB_1] : d_T\,[SB_2]\;. \end{aligned}$$

(iii) The ratios of the lengths of the corresponding segments on the parallel lines are the same. That is,

$$d_T\,[AB] : d_T\,[BC] = d_T\,[A_1B_1] : d_T\,[B_1C_1] = d_T\,[A_2B_2] : d_T\,[B_2C_2]\,.$$

Notice that here $a : b : c = a_1 : b_1 : c_1$ if and only if $a/a_1 = b/b_1 = c/c_1$.

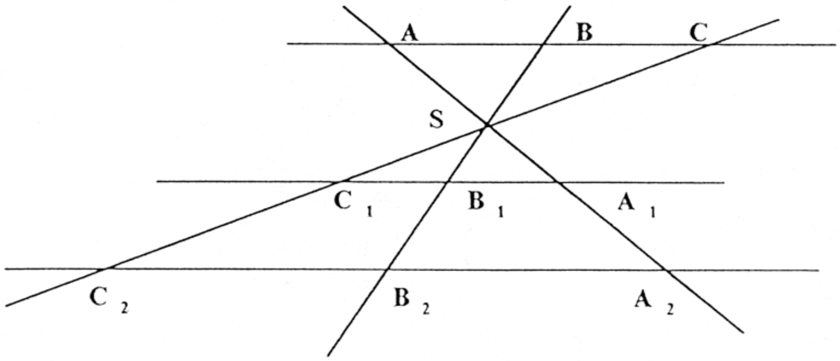

Figure 2.

References

1. Z. Akca and R. Kaya, "On the Taxicab Trigonometry," *Jour. of Inst. of Math. & Comp. Sci. (Math. Ser.),* 10 (1997), 151–159.

2. C. Ekici, I. Kocayusufoglu and Z. Akca, "The Norm in Taxicab Geometry," *Tr. J. of Mathematics,* 22 (1998), 295–307.

3. Y. P. Ho and Y. Liu, "Parabolas in Taxicab Geometry," *Missouri J. of Math.,* 8 (1996), 63–72.

4. R. Kaya, Z. Akca, I. Günaltitli and M. Özcan, "General Equation for Taxicab Conics and Their Classification," *Mitt. Math. Ges. Hamburg,* 19 (2000), 135–148.

5. E. F. Krause, *Taxicab Geometry,* Addison–Wesley, Menlo Park, California, 1975.

6. H. Kreul, K. Kulke, H. Pester, R. Schroedter, *Mathematik Leicht Gemacht,* Verlag Harri Deutsch–Thun–Frankfurt am Main, 1990.

7. R. Laatsch, "Pyramidal Sections in Taxicab Geometry," *Mathematics Magazine,* 55 (1982), 205–212.

8. K. Menger, *You Will Like Geometry,* Guildbook of Illinois Institute of Technology Geometry Exhibit, Museum of Science & Industry, Chicago, Illinois, 1952.

9. H. Minkowski, *Gesammelte Abhandlungen,* Chelsea Publishing Co., New York, 1967.

10. B. E. Reynolds, “Taxicab Geometry,” *Pi Mu Epsilon Journal,* 7 (1980), 77–88.

11. D. J. Schattschneider, “The Taxicab Group,” *Amer. Math. Monthly,* 91 (1984), 423–428.

12. S. S. So and Z. S. Al–Maskari, “Two Simple Examples in Non–Euclidean Geometry,” *Kansas Science Teacher (Journal of Mathematics and Science Teaching),* 11 (1995), 14–18.

13. K. O. Sowell, “Taxicab Geometry – A New Slant,” *Mathematics Magazine,* 62 (1989), 238–248.

14. S. Tian, S. S. So and G. Chen, “Concerning Circles in Taxicab Geometry,” *Int. J. Math. Educ. Sci. Technol.,* 28 (1997), 727–733.

Münevver Özcan
Department of Mathematics
Faculty of Science and Arts
University of Osmangazi
Eskişehir – TURKEY
email: mozcan@ogu.edu.tr

Rüstem Kaya
Department of Mathematics
Faculty of Science and Arts
University of Osmangazi
Eskişehir – TURKEY
email: rkaya@ogu.edu.tr