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CONNECTIONS AND MINIMIZING GEODESICS OF TAXICAB GEOMETRY

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ABSTRACT : The aim of this paper is to define the principle fibre bundle and connections and point out the minimizing geodesics of the taxicab geometry.

1. INTRODUCTION

Taxicab geometry is a non-Euclidean geometry. It is defined in 1975 by E. F. Krause by using the taxicab metric

$$d_T(A,B) = |a_1 - b_1| + |a_2 - b_2|$$

for $A = (a_1, a_2)$, $B = (b_1, b_2) \in R^2$.

It is very close to Euclidean geometry, only the distance functions are different. It is a better model for urban world [4].

Let us denote taxicab plane by R_T^2 . Considering R_T^2, we will discuss the idea of principal fibre bundle and connections of taxicab geometry. We first start with recalling the basic definitions. We note that the main reference of this section is [1].

It is proved in [3] that the isometries of taxicab geometry are all translations, $T(2)$, and the orthogonal group $O_T(2) = S \cup R_\theta$, consisting of four reflections and four rotations, defined below.

TRANSLATIONS :

$$T_a : R_T^2 \to R_T^2 \ni T_a(P) = a + P \ \text{ for every } \ a \in R, \ \ P \in R_T^2$$

REFLECTIONS :

$$S = \{(x, y) \ \ | \ \ x = 0, \ \ y = 0, \ \ y = x \text{ and } y = -x\} \ .$$

ROTATIONS :

$$R_\theta = \left\{ A_\theta \ \ | \ \ \theta = k\frac{\pi_T}{2}, \ \ k = 0, 1, 2, 3 \ ; \ \pi_T = 4 \right\}$$

The main subject of this paper is to give relations between the geodesics of taxicab plane which we call them di-piecewise segments and the connection which we call taxicab connection, defined below. Although there is a metric, taxicab metric, on the taxicab plane, it is not induced from a Riemannian metric [3]. Thus, taxicab connection is not a Riemannian connection. However, taxicab connection, even defined on trivial bundle and locally looks like the Euclidean connection, it is quite different from it.

Definition 1 *Let M be a manifold and G a Lie group. A (differentiable) principal fibre bundle over M with structure group G consists of a manifold P and an action of G on P satisfying the following conditions:*

(1) G acts freely on P on the right.

(2) M is a quotient space of P by the equivalence relation induced by G, $M = P/G$, and a canonical projection $\pi : P \to M$ is differentiable.

(3) P is locally trivial.

We denote the principal fibre bundle by $P(M, G)$. We call P the total space, M the base space, G the structure space, and π is the projection mapping. For each $x \in M$, $\pi^{-1}(x)$ is a submanifold of P, called the fibre over x [1].

Example 1 *(Trivial Fibre Bundle): Let G be a Lie group and M a manifold. Let $P = M \times G$ and define the right action on M as follows.*

$$\begin{array}{rlcl} R: & (M \times G) \times G & \longrightarrow & M \times G \\ & ((x, a), b) & \longrightarrow & (x, ab) \end{array}$$

Then, we obtain a differentiable principal fibre bundle $(M \times G)$ (M, G) over the base manifold M with structure group G which is called trivial fibre bundle over M with the structure group G.

As in [1], let $P(M, G)$ be a principal fibre bundle over a manifold M with the structure group G. For each $u \in P$, let $T_u(P)$ be the tangent space of P at u and G_u the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through u. A *connection* Γ in P is an assignment of a subspace Q_u of $T_u(P)$ to each $u \in P$ such that

(a) $T_u(P) = G_u \oplus Q_u$

(b) $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in G$, where R_a is the transformation of P (right action) induced by $a \in G$, $R_a u = ua$

(c) Q_u depends differentiably on u.

Condition (b) means that the distribution $u \to Q_u$ is invariant by G. We call G_u the *vertical subspace* and Q_u the *horizontal subspace* of $T_u(P)$. A vector $X \in T_u(P)$ is called *vertical* (resp. *horizontal*) if it lies in G_u (resp. Q_u). By (a), every vector $X \in T_u(P)$ can be uniquely written as

$$X = Y + Z \quad \text{where } Y \in G_u \quad \text{and} \quad Z \in Q_u \; .$$

We call Y (resp. Z) the *vertical* (resp. *horizontal*) component of X and denote it by vX (resp. hX). Condition (c) means, by definition, that if X is a differentiable vector field on P so are vX and hX. It can be easily verified that this is equivalent to saying that the distribution $u \to Q_u$ is differentiable.

2. TAXICAB LIE GROUP : ($\mathbf{R}_T(2)$).

Consider $A \in O_T(2)$ and define $U_A = \{f \mid f = T_C \circ A,\ T_C \in T(2)\}$. It is clear that $K = \{U_A \mid A \in O_T(2)\}$ is a covering of taxicab isometries $R_T(2)$. The function $x_A : R_T(2) \to R^2$ such that $x_A(T_C \circ A) = (u, v)$ is a chart of $R_T(2)$ where $C = (u, v) \in R^2$. These charts x_A, $A \in O_T(2)$, define a C^∞ structure on $R_T(2)$ and therefore $R_T(2)$ becomes a C^∞ 2-manifold with this structure. On the other hand, the group operation on $R_T(2)$, denoted by Θ,

$$\Theta : R_T(2) \times R_T(2) \to R_T(2)$$

such that

$$\Theta(T_{C_1} \circ A_1, T_{C_2} \circ A_2) = T_{C_1 + A_1 C_2} \circ A_1 A_2$$

is differentiable since the coordinate representation $\Psi : R^4 \to R^2$ of Θ with respect to the charts x_A, x_B and x_{AB} can be written as

$$(u_1, v_1, u_2, v_2) \to (u_1 + au_2 + bv_2, v_1 + cu_2 + dv_2)$$

where the orthogonal isometry A_1 has the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in R$ are constants. With respect to the chart x_A, the function

$$\nu : R_T(2) \to R_T(2)$$

such that

$$\nu(T_C \circ A) = (T_C \circ A)^{-1} = T_{-A^{-1}C} \circ A^{-1}$$

has the coordinate representation

$$(u, v) \longrightarrow -\frac{1}{\det(A)}(du - bv, -cu + av).$$

Thus, the function ν is differentiable. As a result, we can give the following definition.

Definition 2 *We call the isometries group $R_T(2)$ of taxicab plane with the C^∞structure defined by the charts x_A, $A \in O_T(2)$ by Taxicab Lie Group.*

Notice that the Taxicab Lie Group has eight connected components.

3. TAXICAB BUNDLE AND TAXICAB CONNECTION

Let $M = R_T^2$, $G = R_T(2) = T(2) \cdot O_T(2)$, $P = M \times G = R_T^2 \times R_T(2)$ be the base manifold, the structure group and the fibre manifold, respectively. Let us define an action on P

$$\begin{array}{rlcl} R : & P \times R_T(2) & \longrightarrow & P \\ & (((x, y), A), B) & \longrightarrow & ((x, y), AB) \end{array}$$

It is easy to see that R acts from the right on P and is a free action. So, we have the following theorem.

Theorem 1 *$R_T^2 \times R_T(2)$ $(R_T^2, R_T(2))$ is a trivial fibre bundle over R_T^2 with the structure group $R_T(2)$.*

Definition 3 *The trivial fibre bundle $R_T^2 \times R_T(2)$ $(R_T^2, R_T(2))$is called the taxicab bundle over taxicab plane.*

Now, we will introduce a connection on the taxicab bundle. We begin with determining the vertical subspaces of taxicab bundle. For this, let $f = T_{C_0} \circ A \in R_T(2)$ be an isometry of R_T^2 and $u = ((x,y), T_{C_0} \circ A) \in P$. So, we have

$$\begin{aligned} uG &= uR_T(2) \\ &= \{((x,y)\ ,(T_{C_0} \circ A) \circ (T_{C_1} \circ B)) \ \mid\ T_{C_1} \circ B \in R_T(2)\} \\ &= \{((x,y)\ ,T_{C_0+AC_1} \circ AB) \ \mid\ \forall\, B \in O_T(2),\ \ \forall\, C_1 \in R_T^2\} \\ &= \{(x,y)\} \times R_T(2) \end{aligned}$$

Since the equation

$$((x,y), (T_{C_0} \circ A) \circ (T_X \circ B)) = ((x,y),\ T_C \circ H)$$

has a unique solution $X \in R_T^2$, $B \in O_T(2)$ for arbitrary $H \in O_T(2)$ and $C \in T(2)$, we have

$$G_u = T_u(uG) = T_f R_T(2)$$

Then, if we choose

$$Q_u = T_{(x,y)} R_T^2$$

it follows that

$$T_u P = G_u \oplus Q_u$$

Claim :

$$R_g Q_u = Q_{ug}$$

for every $g = T_{C_0} \circ A \in R_T(2)$.

Proof of Claim :

Since

$$\begin{aligned} R_g :\ P &\longrightarrow P \\ u &\longrightarrow ug \end{aligned}$$

it follows that

$$\begin{array}{rlll} R_g: & R_T^2 \times R_T(2) & \longrightarrow & R_T^2 \times R_T(2) \\ & ((x,y), T_C \circ B) & \longrightarrow & ((x,y), (T_C \circ B) \circ (T_{C_0} \circ A)) \end{array}$$

Thus, the restriction of R_g on R_T^2 is the identity map, that is

$$R_g \mid_{R_T^2} = id$$

and therefore

$$\begin{aligned} R_g(Q_u) &= R_g\left(T_{(x,y)} R_T^2\right) \\ &= id\left(T_{(x,y)} R_T^2\right) \\ &= T_{(x,y)} R_T^2 \end{aligned}$$

Finally,

$$R_g Q_u = Q_{ug}$$

as claimed.

It is clear that $u \to Q_u$ is differentiable. As a result, we can conclude that

$$\Gamma : u \longrightarrow \Gamma(u) = Q_u = T_{(x,y)} R_T^2$$

is a connection on $P = R_T^2 \times R_T(2)$. Taking account of the notations that just introduced here, we can give the following definition:

Definition 4 *We call the connection*

$$\Gamma : u \longrightarrow \Gamma(u) = Q_u = T_{(x,y)} R_T^2$$

on the taxicab bundle $R_T^2 \times R_T(2)$ $(R_T^2, R_T(2))$ *by the taxicab connection.*

4. MINIMIZING GEODESICS OF TAXICAB PLANE

Of course like for all connections, the taxicab connection satisfies that the projection map

$$\pi : Q_u \to T_{(x,y)}R_T^2$$

is a linear isomorphism which makes the horizontal spaces as the taxicab plane (since $T_{(x,y)}R_T^2$ can be regarded as the taxicab plane R_T^2).

On the other hand, taxicab connection can be regarded as the restricted Euclidean connection since all connected components of the fibre bundle $R_T^2 \times R_T(2)$ (there are eight of all) are diffeomorphic to $R_T^2 \times T(2)$. Thus, Christtoffel symbols are all zero with respect to the local basis. In this view of point, the geodesics on the taxicab plane are line segments or broken line segments. Therefore, the geodesics are the same as the geodesics of the Euclidean plane. But there is a point that makes all things different from the Euclidean case which is the answer of the following question.

"*Which geodesics of the taxicab plane are length minimizing?*"

The answer comes up as a theorem by using the notion of di-piecewise segments.

Definition 5 *Let $\alpha : [a,b] \to R_T^2$ be a curve. We call the curve α di-piecewise segment if there exists a partition $P = (x_1, x_2, ..., x_n)$ of the interval $[a,b]$ and all restrictions α_i of α to the subintervals $[x_i, x_{i+1}]$, $1 \leq i < n$, are line segments satisfying the following property: If any α_i, $1 \leq i < n$, is increasing (decreasing), then all others are increasing (decreasing) in the sense of the graph of α_i is increasing (decreasing) as in the Euclidean plane R^2*

It is clear that if the points A and B are on a given di-piecewise segment, then the distance $d_T(A, B)$ is equal to the arc-length between the points A and B of the given di-piecewise segment.

Theorem 2 *Length minimizing geodesics of the taxicab plane are di-piecewise segments.*

Proof. It is clear that di-piecewise segments are geodesics on the taxicab plane. Let $\alpha : I \to R^2_T$ be a di-piecewise segment from the point $\alpha(0) = p$ to the point $\alpha(1) = q$. We can write, by [2],

$$\begin{aligned} L_p^q(\alpha) &= \textstyle\sum_{i=1}^{n} \int_0^1 \| \alpha_i'(t) \| \, dt \\ &= \textstyle\sum_{i=1}^{n} \int_0^1 \| p_{i+1} - p_i \| \, dt \\ &= d_T(p, q) \end{aligned}$$

where $\alpha_i(t) = p_i + t(p_{i+1} - p_i)$ is the segment of α . Thus, the arc-length of di-piecewise segment between a given two points is equal to the distance between them.

We, now, must prove that there is no geodesic, β, from a given point p to a point q such that the arc-length of β is less than the arc-length of any di-piecewise segment.

Suppose there is one such geodesic, say β. Then, the arc-length of β from the point p to the point q, $L_p^q(\beta)$, satisfies the inequality

$$L_p^q(\beta) \leq L_p^q(\alpha) = d_T(p, q).$$

Since d_T is a metric, it follows that

$$L_p^q(\beta) = L_p^q(\alpha) = d_T(p, q).$$

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